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*LECTURES ON THE*  
*CALCULUS OF VARIATIONS*

(The Weierstrassian Theory)

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## PREFACE.

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Since the time of Newton and the Bernoullis, problems have been solved by methods to which the general name of the Calculus of Variations has been applied. These methods were generalized and systematized by Euler, Lagrange, Legendre and their followers; but numerous difficulties arose. Some of these were removed by Jacobi and his contemporaries. Still many of the methods had to be extended and it was necessary to supply much that was deficient and to make clear what remained obscure. The progress of Analysis is indebted to the genius of Weierstrass for the perfection of this theory.

While a student in the University of Berlin it was my privilege to hear the lectures of Professor H. A. Schwarz on the Calculus of Variations. In its presentation this eminent mathematician followed his great teacher, Weierstrass, who had established the theory on a firm foundation, free from objection, simple and at the same time more comprehensive than it had been hitherto. I also took the opportunity to study Weierstrass's lectures, of which there were copies in the *Mathematischer Verein*.

Through the courtesy of Professor Ormond Stone abstracts of this theory were published in Volumes IX, X, XI and XII of the *Annals of Mathematics*, and from time to time the Calculus of Variations has been included in my University lecture courses.

I have delayed the publication of these lectures with the hope that Weierstrass's lectures would be published by the commission

to which has been intrusted the editing of his complete works. This publication, however, seems remote, and commentaries on the Calculus of Variations are becoming so numerous that I have deemed it expedient to bring out my work at the present time.

As one would naturally expect, I have followed Weierstrass's treatment of the subject; in many places, especially in the latter part of the book, my lectures are little more than a repetition of his. It is from the Weierstrassian standpoint that I have developed my own ideas and have presented those derived from other writers. Thus, instead of giving separate accounts of Legendre's and Jacobi's works introductory to the general treatment, I have produced their discoveries in the proper places in the text, and I believe that by this means confusion has been avoided which otherwise might be experienced by students who are reading the subject for the first time. I hope that this exposition of the fundamental principles may prove attractive. The reader will then naturally desire a more extensive knowledge regarding the literature and the various improvements that have been made by successive mathematicians. He will wish to follow the methods which they have employed, and will seek further information regarding the historical development. References are given on Pages 18 and 19 of the text from which the original sources are easily obtained.

The necessary and sufficient conditions as they arise for the existence of a maximum or a minimum are illustrated by six problems, which are worked out step by step in the theory. They have been chosen to represent the different phases of the subject, the exceptional cases which may occur, the discontinuous solutions, etc. For example, in the first problem it is found that a minimum may be offered by an irregular curve, whereas seemingly

the problem is satisfied by a regular curve, the *catenary*. Attention is thereby called to the fact that although our integrals have a meaning only when taken over regular curves, we have to guard against discontinuous solutions, and consequently further conditions for the existence of a maximum or a minimum must be derived. The case of the discontinuous solution is considered in this problem as also when the limits of integration are two *conjugate points*. Newton's problem is introduced to show that one of the necessary conditions is not satisfied and that there is no curve which fulfills the given requirements.

By the formulation of such problems in Chapter I we come readily to the statement of the general problem of the Calculus of Variations.

In the general discussion attention has been confined for the most part to the realm of two variables, and in this realm only the first derivatives of the variables have been admitted. Generalizations and extensions are suggested which, as a rule, may be executed with little difficulty.

The second part of the work beginning with Chapter XIII treats of the theory of *Relative Maxima and Minima*, where the isoperimetrical problems are considered. Here also the existence of a *field* about the curve which is to maximize or minimize a given integral is emphasized and the necessary and sufficient conditions are derived and proved in a manner similar to that by which the analogous conditions are found in the first part of the work.

My wish in these lectures has been to give a connected and simple treatment of what may be called the *Weierstrassian Theory of the Calculus of Variations*. Many instructive theo-

remains of older writers have been omitted. I regret too that there has not been room to take up some of the investigations which have recently appeared. It is seldom that the first edition of a book is the final form in which an author wishes to leave his work. As I expect to make additions and alterations in my University lectures from time to time, I shall receive with pleasure any suggestions that may be offered.

In conclusion, I wish to take the opportunity here of returning my sincere thanks to the Board of Directors of the University of Cincinnati for their liberality in the publication of this work.

My thanks are also due to Mr. Harold P. Murray, Manager of the University Press, for his careful supervision of the printing.

HARRIS HANCOCK.

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## CHAPTER I.

### PRESENTATION OF THE PRINCIPAL PROBLEMS OF THE CALCULUS OF VARIATIONS.

1. At the time when the Differential Calculus, and in part also the Integral Calculus, were being formulated, certain problems were proposed, which, although not belonging to the province of the Theory of Maxima and Minima, had a marked semblance to the problems of that theory, and were often solvable by methods belonging to it. The following was one of the first problems proposed:

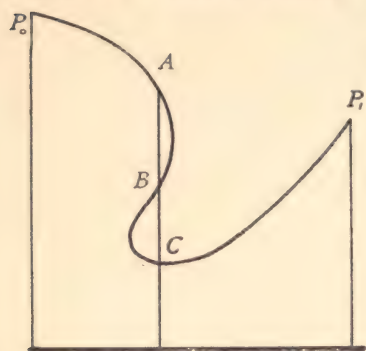
PROBLEM I. *Two points  $P_0$  and  $P_1$  with coordinates  $(x_0, y_0)$  and  $(x_1, y_1)$  respectively are given. Both points lie on the same side of the axis of  $X$  in the plane- $xy$ . It is required to join  $P_0$  and  $P_1$  by a curve which lies in the upper half of the  $xy$ -plane (axis of  $X$  inclusive) such that when the plane is turned through one complete revolution about the axis of  $X$ , the zone generated by this curve may have the smallest possible surface-area.*

We may use this problem to illustrate the connection between the Calculus of Variations and the Theory of Maxima and Minima; at the same time the difference between the two theories is evident.

2. If we try to solve the problem of the preceding article by the methods of the Theory of Maxima and Minima, we must proceed as follows:

Suppose that it is possible to draw a curve between  $P_0$  and  $P_1$  which satisfies the problem. Then every portion of this curve, however small, must have the property of generating a surface of smallest area. For, suppose a change is made in an arbitrary portion of the curve, however small, and let the remaining portion of the curve be unchanged. If by this change the surface-area generated by this arbitrary portion of curve is less than it was before,

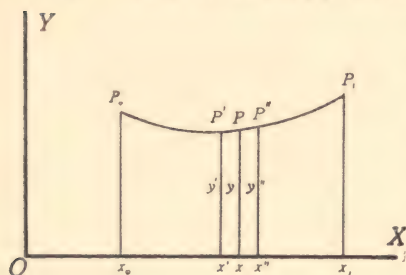
then the curve containing the deformed portion of curve generates a smaller surface-area than the original curve. Also, if to one value of  $x$  there belong several values of  $y$ , then, instead of the



portion of curve belonging to the same abscissa, we might take the straight line which joins these two points. This line would generate a surface of smaller area than that generated by the curve that passes through the same two points. Hence the curve would generate a surface which did not have a minimum area. We may therefore consider the curve as divided into portions such

that the projections of these portions on the axis of  $X$  are all equal.

3. The above hypotheses being granted, we suppose that the



two points  $P' (x', y')$  and  $P'' (x'', y'')$  are taken on the curve, and we find another point  $P (x, y)$  on the curve such that  $x - x' = x'' - x = \Delta x$ . We suppose that  $P$  and  $P'$ ,  $P$  and  $P''$  are joined together by straight lines, and later we suppose that these straight

lines are taken so close together that there is a transition from the straight lines to the curve. The remaining portions of curve on the left-hand side of  $P'$  and on the right-hand side of  $P''$  are supposed to remain unaltered.

The portions of surface-area generated by the straight lines  $P'P$  and  $PP''$  are

$$\pi (y' + y) \sqrt{(\Delta x)^2 + (y - y')^2} \text{ and } \pi (y + y'') \sqrt{(\Delta x)^2 + (y'' - y)^2}.$$

In order to have a minimum the sum of these two expressions when differentiated with regard to  $y$  must be zero; *i. e.*,

$$\begin{aligned} & \pi \sqrt{(\Delta x)^2 + (y - y')^2} + \pi \sqrt{(\Delta x)^2 + (y'' - y)^2} \\ & + \frac{\pi (y' + y) (y - y')}{\sqrt{(\Delta x)^2 + (y - y')^2}} - \frac{\pi (y + y'') (y'' - y)}{\sqrt{(\Delta x)^2 + (y'' - y)^2}} = 0 \dots \dots [A] \end{aligned}$$

The quantity  $y$  may be determined from this equation as a func-

tion of  $x$ , so that  $y=f(x)$ , say. We therefore have  $y'=f'(x-\Delta x)$  and  $y''=f'(x+\Delta x)$ . Hence by Taylor's Theorem,

$$\begin{aligned} y' &= f'(x-\Delta x) = f'(x) - f''(x) \Delta x + \frac{1}{2} f'''(x) (\Delta x)^2 - \dots, \\ y'' &= f'(x+\Delta x) = f'(x) + f''(x) \Delta x + \frac{1}{2} f'''(x) (\Delta x)^2 + \dots; \end{aligned}$$

and consequently,

$$\begin{aligned} y - y' &= f'(x) \Delta x - \frac{1}{2} f''(x) (\Delta x)^2 + \dots, \\ y'' - y &= f'(x) \Delta x + \frac{1}{2} f''(x) (\Delta x)^2 + \dots \end{aligned}$$

Substituting these values in  $[A]$ , we have, neglecting the factor  $\pi$ ,

$$\begin{aligned} & \Delta x \sqrt{1 + f'(x)^2} - f'(x) f''(x) \Delta x + \dots \\ & + \Delta x \sqrt{1 + f'(x)^2} + f'(x) f''(x) \Delta x + \dots \\ & + \frac{[2f(x) - f'(x) \Delta x + \dots][f'(x) \Delta x - \frac{1}{2} f''(x) \Delta x + \dots]}{\Delta x \sqrt{1 + f'(x)^2} - f'(x) f''(x) \Delta x + \dots} \\ & - \frac{[2f(x) + f'(x) \Delta x + \dots][f'(x) \Delta x + \frac{1}{2} f''(x) \Delta x + \dots]}{\Delta x \sqrt{1 + f'(x)^2} + f'(x) f''(x) \Delta x + \dots} = 0. \end{aligned}$$

Expand this expression in ascending powers of  $\Delta x$ , divide through by  $\Delta x$  and then make  $\Delta x = 0$ . We then have

$$1 + f'(x)^2 - f(x) f''(x) = 0;$$

$$\text{or} \quad 1 + \left(\frac{dy}{dx}\right)^2 - y \frac{d^2y}{dx^2} = 0 \dots \dots [B].$$

Therefore in order to have a minimum value,  $f(x)$  or  $y$  must satisfy this differential equation; however, when  $y$  satisfies this differential equation we do not always have a minimum, as will be shown later.

In other words, the differential equation  $[B]$  is a necessary consequence of the supposed existence of a minimal surface of revolution. As a condition, however, it is not sufficient to assure the existence of a curve giving such a surface.

Differentiate the equation  $[B]$  with regard to  $x$ , and we have

$$\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = y \frac{d^3y}{dx^3},$$

or

$$\frac{\frac{dy}{dx}}{y} = \frac{\frac{d}{dx} \left( \frac{d^2y}{dx^2} \right)}{\frac{d^2y}{dx^2}}.$$

Integrating, we have

$$y = c^2 \frac{d^2y}{dx^2},$$

where  $c^2$  is the constant of integration. Since  $y = e^{\frac{x}{c}}$  and  $y = e^{-\frac{x}{c}}$  are two solutions of this last differential equation, the general solution is

$$y = c_1 e^{\frac{x}{c}} + c_2 e^{-\frac{x}{c}},$$

where  $c_1$  and  $c_2$  are constants. This last equation is that of the *catenary* curve.

4. Thus, by the help of the Theory of Maxima and Minima, we have, it is true, come to a certain result; but, on the other hand, we have yet to ask whether this curve gives a true minimum; and owing to the manner in which we have arrived at these conclusions, we have yet to see whether this curve only in a definite portion or throughout its whole extent possesses the property required in the problem.

That we are justified in insisting upon this last statement is seen from what follows later, where it will be shown that the curve found above satisfies the required conditions only between given limits.

A simple consideration shows that the method we have followed above is not at all rigorous; since it presupposes, which of itself is not admissible, that the curve which satisfies the problems is regular in its whole extent, for otherwise the portions of curve between the two points  $(x - \Delta x, y')$  and  $(x, y)$  could not be replaced by straight lines joining these two points; also, the expansion by Taylor's Theorem would not have been admissible.

5. The characteristic difference between problems relative to Maxima and Minima and the problems which have to do with

the Calculus of Variations consists in the fact that, in the first case, we have to deal with only a *finite number of discrete points*, while in the Calculus of Variations, the question is concerning a *continuous series of points*.

If we wish to substitute in the place of the curve first a polygonal line and afterwards apply to this line methods similar to those used above, then it turns out that, after we have found a line which satisfies all the conditions, it is necessary yet to prove that the required limiting transition from polygonal line to curve *in reality* results in a definite curve which satisfies the conditions of the problem.

6. Every limiting transition, as from polygon to curve, is made of itself, if we make use of the conception of integration, since an integral represents the limiting value of a sum of quantities which, following a definite law, increase so as to become infinite in number, the quantities themselves becoming smaller in a corresponding manner.

If we therefore define the surface-area of the curve  $y=f(x)$ , which we have to find, by

$$S = 2\pi \int y \, ds,$$

or

$$\frac{S}{2\pi} = \int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx,$$

then this integral will have a definite value for every curve that is drawn between  $P_0$  and  $P_1$ , and consequently the problem may be stated as follows:

PROBLEM I. *y is to be so determined as a function of x that the above integral shall have the smallest possible value.*

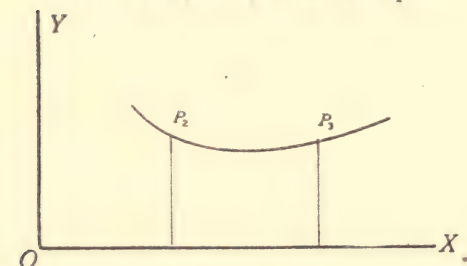
The solution of this problem will be given later. The two methods given above have been chosen to make clear what there is in common in the Theory of Maxima and Minima and the Calculus of Variations, and also to show the difference between them.

In the *Differential Calculus* a definite function is given, and a special value of the variable or variables (if there are more than one variable) is sought, for which the function takes the greatest

or least possible value; in the *Calculus of Variation* a function is sought and an expression is given which depends upon this function in a certain known manner. A definite integral is considered, in which the integrand depends upon the unknown function in a known manner, and it is asked what form must the unknown function have in order that the definite integral may have a maximum or a minimum value.

We treat only real values of the variables.

7. If  $t_2 < t_3$  and the point  $P_2$  corresponds to  $t_2$  and  $P_3$  to  $t_3$ , then  $P_3$  with reference to  $P_2$  is known as a *later* point; and  $P_2$  with reference to  $P_3$  is known as an *earlier* point.



As was shown in Art. 2, the ordinate  $y$  of the required curve is a one-valued function of the abscissa  $x$ . It often happens that one cannot know *à priori* that one of the ordinates is a one-valued function of the other. Poincaré\* has shown that it is always possible to express the two variables  $x, y$ , when there is an analytic relation between them, as one-valued functions of a third variable  $t$ . The only property that is required of this variable is, when it traverses all values between two given limits, the corresponding point  $(x, y)$  traverses the curve from the initial point to the end point, and in such a way that for a greater value of  $t$  there belongs a later point of the curve.

For example, suppose that  $z = x^y$  where  $x$  and  $y$  are two independent variables. Then in virtue of this equation there is no way of expressing the dependence of one of these variables upon the other without the introduction of transcendental functions. But if we write

$$x = e^t,$$

then

$$z = e^{y^t}$$

or

$$y = \frac{\log z}{t}.$$

\* Poincaré (Bulletin de la Société Mathématique de France, T XI. 1883.) See also my lectures on the Theory of Maxima and Minima, etc. Page 13.

Thus  $x$  and  $y$  are one-valued functions of the variable  $t$ .

If then we introduce such a new variable  $t$  in the integral of Problem I, that integral becomes

$$\frac{S}{2\pi} = \int_{t_0}^{t_1} y \sqrt{x'^2 + y'^2} dt,$$

where we denote by  $x'$  and  $y'$  the quantities  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ .

We may now state Problem I as follows:

*The quantities  $x$  and  $y$  are to be determined as one-valued functions of a parameter  $t$  in such a way that the above integral will have the smallest possible value.*

8. That we may learn the essential properties of the Calculus of Variations, we shall next formulate other simple problems; then, while we seek the general characteristics of these problems, we shall of our own accord come to a more exact statement of the problems which the Calculus of Variations has to solve.

As a second problem may be given the very celebrated problem of the Calculus of Variations, that of the *brachistochrone*\* (curve of quickest descent), which may be stated as follows:

PROBLEM II. *Two points  $A$  and  $B$  are situated in a vertical plane, the point  $B$  being situated lower than the point  $A$ ; a curve is to be drawn between these points in such a manner that a material point subject to the action of gravity and compelled to move upon this curve with a given initial velocity, shall go from the point  $A$  to the point  $B$  in the shortest possible time.*

Let the mass of the material point be 1, its initial velocity  $a$ , the acceleration of gravity  $2g$ , the time  $t$ , and the coordinates of

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\* Woodhouse (A Treatise on Ipsoperimetrical Problems and the Calculus of Variations, 1810) writes (p. 1): "The ordinary questions of maxima and minima were amongst the first that engaged the attention of mathematicians at the time of the invention of the Differential Calculus (1684), three years before the publication of the Principia. The first problem relative to a species of maxima and minima distinct from the ordinary was proposed by Newton in the Principia; it was that of the *solid of least resistance*. But the subject became not matter of discussion and controversy till John Bernoulli (Acta Erudit., 1696, p. 269) required the *curve of quickest descent*."

$A$  and  $B$  respectively  $(o, o)$  and  $(a, b)$ . Let the direction of the positive  $Y$ -axis be the direction of a falling body (due to gravity) and let the positive  $X$ -axis be directed toward the side on which the point  $B$  lies. Then, according to the law of the *Conservation of Energy*,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4gy + a^2,$$

or,

$$dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{4gy + a^2}} = \frac{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}{\sqrt{4gy + a^2}} dy;$$

whence

$$T = \int_0^b \frac{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}{\sqrt{4gy + a^2}} dy.$$

We have then as our problem: *so determine  $x$  as a function of  $y$  that the above integral shall have the smallest possible value.*

As regards the signs of the roots that appear in the above integral, it is evident that these signs must be the same at the beginning of the motion and may be taken positive. For on mechanical grounds it follows that the curve must at first descend; consequently at the beginning of the motion  $y$  increases with increasing  $t$ , and is therefore positive. Since  $4gy + a^2$  is always a positive quantity, being equal to  $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ , and can never vanish, we may always give to  $\sqrt{4gy + a^2}$  the positive sign. Also at the beginning of the motion the quantity  $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$  must have the positive sign, since  $dt$  always represents a positive increment of time. However, in the further course of the motion, it may happen that  $dy = 0$ . Then the quantity  $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$  passes through infinity, so that  $dy$  and  $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$  may simultaneously change their sign, while  $\sqrt{4gy + a^2}$  continues with the positive sign.

9. The assumption made in the statement of the problem that  $B$  must lie below  $A$  is not essential. For the material point has at  $B$  a certain velocity  $\beta$ , which we may calculate from the initial velocity  $\alpha$  and the height of  $A$  above  $B$ . When the point reaches  $B$  with this velocity it may rise again, and it will have the original velocity when it has reached the height  $A$  on the other side of  $B$ . The time which is necessary for the ascent is the same as that required in the descent, if we assume that the curve along which the ascent takes place is symmetrical with that of the descent.

If, therefore, the point started from  $B$ , we could calculate from  $\beta$ , which is now the initial velocity, the velocity  $\alpha$  at the point  $A$ . We then have the curve in question, if we seek the curve along which the point with the initial velocity  $\beta$  reaches  $A$  in the shortest time.

In the case of the present problem we see from physical considerations that  $y$  is a one-valued function of  $x$ . As this is not possible in all cases, it is expedient to represent the curve here also by two equations; that is, to consider  $x$  and  $y$  as one-valued functions of a third variable  $t$ ,\* where  $t$  is subject to the only condition, that when it goes through all values between two given limits, the corresponding point  $x, y$  traverses the curve from the beginning-point to the end-point and in such a way that to a greater value of  $t$  there corresponds a later point of the curve.

The above integral becomes

$$T = \int_{t_0}^{t_1} \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + \alpha^2}} dt,$$

where we have written  $x'$  and  $y'$  for  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  respectively.

Our problem then is: *Determine  $x$  and  $y$  as functions of a parameter  $t$  in such a way that the integral just written may have the smallest possible value.*

10. PROBLEM III. *Between two points on a regular surface  $f(x, y, z) = 0$ , a curve is to be drawn so that its length is a minimum.*

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\* This  $t$  is, of course, different from the time  $t$  of the preceding article.

Consider the orthogonal coordinates  $x, y, z$  of a surface represented as one-valued regular functions\* of two parameters  $u$  and  $v$ . If we consider these as the rectangular coordinates of a point on the plane, then to every point of the surface there will correspond a definite point of the  $uv$ -plane, and these points in their collectivity fill out a definite portion of the plane, which may be looked upon as the image of the surface on the plane. To every curve on the surface corresponds a curve in this part of the  $uv$ -plane and reciprocally.

Further, consider  $u$  and  $v$  as one-valued functions of a quantity  $t$ ; hence, to every value  $t$  there corresponds a point of the  $uv$ -plane, and therefore, also, in case this point lies in the definite portion of the  $uv$ -plane, there is a corresponding definite point of the surface.

Consequently if  $t_0$  and  $t_1$  are values of  $t$  which correspond to the two fixed points on the surface, then the length of any curve which lies between these two points is determined through

$$L = \int_{t_0}^{t_1} \sqrt{P \left( \frac{du}{dt} \right)^2 + 2Q \frac{du}{dt} \cdot \frac{dv}{dt} + R \left( \frac{dv}{dt} \right)^2} dt,$$

where

$$P = \left\{ \frac{\partial x}{\partial u} \right\}^2 + \left\{ \frac{\partial y}{\partial u} \right\}^2 + \left\{ \frac{\partial z}{\partial u} \right\}^2,$$

$$Q = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v},$$

$$R = \left\{ \frac{\partial x}{\partial v} \right\}^2 + \left\{ \frac{\partial y}{\partial v} \right\}^2 + \left\{ \frac{\partial z}{\partial v} \right\}^2.$$

We have then to determine  $u$  and  $v$  as functions of  $t$ , so that  $L$  is a minimum.

11. In the case of the above problem it is necessary to apply the representation there given, whereas in Problem I and Problem II the expression of  $x$  and  $y$  as one-valued functions of  $t$  may be regarded as *expedient*. In Problem III the variables  $u$  and  $v$

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\* See my lectures on the Theory of Maxima and Minima, etc. Page 31.

*must* be regarded as functions of a third variable. We cannot regard  $v$  as a function of  $u$ , for we know nothing about the trace of the curve. If we wished to regard  $v$  only as a double-valued function of  $u$ , we would even then encounter many difficulties. Hence the requirements *must* be made that  $u$  and  $v$  be so determined as one-valued functions of  $t$ , that the integral in the preceding article be a minimum.

12. PROBLEM IV. *Find the form of the surface of rotation, which, having an axis lying in a fixed direction, offers the least resistance in moving through a liquid in the direction of the axis, it being supposed that the resistance of an element of surface is proportional to the square of the component of velocity in the direction of its normal.*

This problem is due to Newton.\*

It is assumed that the friction between the body and the fluid and that within the fluid itself may be neglected.

Let the  $Y$ -axis be the axis of rotation,  $ds$  an element of the generating curve,  $\theta$  the angle between the normal and the  $Y$ -axis, so that  $\frac{dx}{ds} = \cos \theta$ .

A zone of the surface is therefore given by

$$2\pi x ds = 2\pi x \sqrt{x'^2 + y'^2} dt.$$

The component of velocity in the normal direction is  $v \cos \theta$ , and the resistance in the normal direction which the zone offers, is

$$v^2 \cos^2 \theta \, 2\pi x \sqrt{x'^2 + y'^2} dt.$$

This quantity multiplied by  $\cos \theta$  gives the resistance in the direction of the  $Y$ -axis. We consequently have the required resistance of the body expressed by the integral

$$\frac{R}{2\pi v^2} = \int \frac{xx'^3}{x'^2 + y'^2} dt.$$

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\* Newton, Principia, Book II, prop. 34. Thus Newton was the first to consider a problem in the Calculus of Variations, and his problem involved a discontinuous solution. Solutions of it have been given by Euler and almost all other writers on the Calculus of Variations. We shall see that one of the principal conditions for a minimum (the condition of Weierstrass) is not satisfied, and that there can never be a maximum or a minimum.

Our problem then is to connect two points  $P_1$  and  $P_2$  by a curve so that the zone which it generates about the  $Y$ -axis offers the least resistance. Neglecting the constant factor  $2\pi v^2$ , we have to determine  $x$  and  $y$  as one-valued functions of  $t$  so that the integral

$$R = \int_{t_0}^{t_1} \frac{xx'^3}{x'^2 + y'^2} dt$$

shall be a minimum.

13. That which is common to the four problems stated above consists in the determination of  $x$  and  $y$  as one-valued functions of a quantity  $t$  in such a way that an integral dependent upon them of the form

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt$$

will have the smallest possible value. Here  $t_0$  and  $t_1$  have fixed values so that the corresponding coordinates  $x, y$  of the initial and the final point of the curve are supposed to be known.

$F(x, y, x', y')$  represents a one-valued regular function of the four arguments  $x, y, x', y'$  of which  $x'$  and  $y'$  (since they represent the direction of the tangent to the curve) are to be regarded as unrestricted, while the region of the point  $x, y$  may be either the whole plane or only a continuous portion of it.

14. The condition that  $t_0, t_1$  should have fixed values is not essential; moreover both end-points may move, as in the case of the third problem, if we give it the following form: *Two curves are given on a surface; among all the possible curves between the points of the one curve and the points of the other, that curve is to be found which has the shortest length.* We are accustomed to call this the geodesic distance of two curves.

In order to solve this problem, we must first solve the special Problem III, since, if a curve has the property of being of minimum length such as is required above, it must also retain the same property, if we consider the end-points fixed. Hence from III the nature of the curve must be determined. The variation of the end-

points gives in addition certain special properties which the curve must possess.

For example, the shortest distance between two curves which lie in the same plane is clearly a straight line; through the variation of the end-points it follows that this straight line must be perpendicular to both curves at the same time.

15. Essentially different from the four problems already given is the following:

PROBLEM V. *It is required to draw a closed curve which with a given periphery inscribes the greatest possible area.*

Let  $x$  and  $y$  be one-valued functions of  $t$ , say  $x(t)$  and  $y(t)$ , such that for two definite values  $t_0$  and  $t_1$  of  $t$  the corresponding points  $x, y$  of the curve coincide, and that, if  $t$  goes from a smaller value  $t_0$  to a greater value  $t_1$ , the point  $x, y$  completely traverses the curve in the positive direction. Then twice the area of the surface included by the curve is expressed by the integral

$$I^{(0)} = \int_{t_0}^{t_1} (xy' - yx') dt,$$

and the periphery of the curve is given by the integral

$$I^{(1)} = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2} dt.$$

Our problem then is: *So determine  $x$  and  $y$  as one-valued functions of  $t$  that  $I^{(0)}$  shall have the greatest possible value, while at the same time  $I^{(1)}$  has a given value.*

16. PROBLEM VI. *What form is taken by an indefinitely thin, absolutely flexible, but inextensible thread which is fixed at both ends, if the action of gravity alone acts upon it?*

This problem offers the characteristics of a minimum, for with stable equilibrium the center of gravity must be as low as possible. If the  $Y$ -axis is taken vertical with the direction upward, and if  $S$  denotes the length of the curve, and  $\xi, \eta$  the coor-

dinates of the center of gravity, then  $\eta$  is determined from the equation

$$\eta S = \int_{t_0}^{t_1} y \sqrt{x'^2 + y'^2} dt,$$

where

$$S = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2} dt.$$

The problem may be stated thus: *the variables  $x$  and  $y$  are to be determined as one-valued functions of a quantity  $t$  in such a way that the first of the above integrals has a minimum value, while the second retains a given fixed value.*

17. Problems V and VI are usually classified under the name, *Relative Maxima and Minima*, a term which requires no further explanation. In general they are included in the following problem: *Let  $F^{(0)}(x, y, x', y')$  and  $F^{(1)}(x, y, x', y')$  be two functions of the same character as the function  $F(x, y, x', y')$  of Art. 13. It is required to determine  $x$  and  $y$  as one valued functions of a quantity  $t$  in such a way that the integral*

$$I^{(0)} = \int_{t_0}^{t_1} F^{(0)}(x, y, x', y') dt$$

*has a maximum or a minimum value, while at the same time the integral*

$$I^{(1)} = \int_{t_0}^{t_1} F^{(1)}(x, y, x', y') dt$$

*conserves a given value.*

18. We shall give in the sequel what we believe to be a rigorous treatment of the problems already formulated. The reader may propose for himself natural extensions of what is given; for example, instead of taking two variables, consider an integral having as integrand a function of  $n$  variables. Further, subject these variables to subsidiary conditions and also allow the second and

higher derivatives of the variables with respect to a quantity  $t$  to enter the discussion. Then double integrals which lead to the study of Minimal Surfaces may be treated by methods of variation (see Arts. 175 et seq.).

19. We may define the object of the Calculus of Variations in a still more general manner by the introduction of a fundamental conception, *that of the variation of a curve*. In former times the Calculus of Variations was considered one of the most difficult branches of analysis. It was wrongly thought that the difficulty was in the supposed lack of clearness in the fundamental conceptions, especially in that of the variation of a curve. The difficulties that arise are mostly in other directions.

In the Theory of Maxima and Minima we say that for a definite system of values of the variables the value of a function is a maximum or a minimum, if this value of the function for this system of values is greater or smaller than it is for all the neighboring systems of values.

*We say\* of a function  $f(x)$  of one variable, it has, at a definite position  $x=a$ , a maximum or a minimum value, if this value for  $x=a$  is respectively greater or less than it is for all other values of  $x$  which are situated in the neighborhood of  $|x-a| < \delta$  as near as we wish to  $a$ .*

The analytical condition that  $f(x)$  shall have for the position  $x=a$

$a$  maximum, is expressed by  $f(x) - f(a) < 0$ ;   
 $a$  minimum, is expressed by  $f(x) - f(a) > 0$ ;   
 } for  $|x-a| < \delta$ .

In the same way we say a function  $f(x_1, x_2, \dots, x_n)$  of  $n$  variables has at a definite position  $x_1=a_1, x_2=a_2, \dots, x_n=a_n$ , a maximum or a minimum, if the value of the function for  $x_1=a_1, x_2=a_2, \dots, x_n=a_n$  is respectively greater or smaller than it is for all other systems of values which are situated in the neighborhood  $|x_\lambda - a_\lambda| < \delta_\lambda$  ( $\lambda=1, 2, \dots, n$ ) as near as we wish to the first position.

As here we speak of a neighboring system of values, so also we speak in the Calculus of Variations of curves which lie in the

\*See Lectures on the Theory of Maxima and Minima of Functions of Several Variables, p. 32.

neighborhood of a given curve; and we require that an integral in the case of a minimum should be *less* and in the case of a maximum *greater* when taken over the given curve than for any of the neighboring curves.

In order to fix the conception of a neighboring curve, and to make clear the analogy of the same with the conception of a neighboring system of values, let us consider first, instead of the given curve, a broken line  $A_1 A_2 A_3 \dots A_n$ , and let us cause the same to slide just a little from its original position.

Then in the new position every corner  $B_k$  will correspond to a definite corner  $A_k$  in the old position, and moreover the new position  $B_1 B_2 B_3 \dots B_n$  will be as little different from the old position  $A_1 A_2 A_3 \dots A_n$  as we wish, if we stipulate that the distance between any two corresponding points  $A_k$  and  $B_k$  shall be smaller than any quantity  $\delta$ , where  $\delta$  is as small as we choose. Now, by increasing the number of sides, let the broken line pass into the given curve; then the points  $B_1, B_2, \dots B_n$  will also form a curve which is little different from the first curve, and which we consequently call *neighboring* to the first curve.

Therefore we can say a curve *is neighboring* to another curve, or exists out of another curve through a *variation*\* as small as we choose, if to every point of the latter curve there corresponds a definite point on the former curve, and also the distance between any two corresponding points is smaller than  $\delta$ , where  $\delta$  is as small as we wish.

The geometrical conception of a neighboring curve offers no obscurity. In a similar manner it is easy to see that for every change of the curve there is a corresponding change of the integral

$$\int F(x, y, x', y') dt,$$

and that this change will be indefinitely small when the second curve is neighboring to the first.

This change of the value of the integral must of course be a *continuous, negative* one if the integral is to be a maximum, and a *continuous, positive* one if the integral is to be a minimum.

\*The notion of the variation of a curve was first introduced by Lagrange. He considered the required curve transposed into one that lies indefinitely near it by writing instead of each point  $x, y$  of the curve another point  $x+\xi, y+\eta$ . This operation of transition he called a *variation*.

20. Observing what has just been said, we may formulate the problems of Arts. 13 and 17 as follows:

*The variables  $x$  and  $y$  are to be determined as one-valued functions of a quantity  $t$  in such a way that when we define a curve by the equations  $x=x(t)$ ,  $y=y(t)$ , and cause the curve to vary as little as we wish, the change which thereby takes place in the integral*

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt$$

*must be continuously positive if a minimum is to enter, and continuously negative if we require a maximum.*

In the case of Relative Maxima and Minima, for every indefinitely small variation of the curve for which the integral

$$I^{(1)} = \int_{t_0}^{t_1} F^{(1)}(x, y, x', y') dt$$

*conserves its value unchanged, the integral*

$$I^{(0)} = \int_{t_0}^{t_1} F^{(0)}(x, y, x', y') dt,$$

*according as to whether a maximum or a minimum is to be present, must be constantly smaller or constantly greater than for the curve which is given by the equations  $x=x(t)$ ,  $y=y(t)$ .*

21. We must seek strenuous methods for the solution of the problems presented above. The methods by means of which Jacobi and the older mathematicians, Bernoulli and his contemporaries, Newton and Leibnitz, sought to solve these questions lead only to the formation of certain differential equations and in propitious cases to the integration of such equations. But these methods were not sufficient for a definitive determination as to whether the curve which had been found *in reality* offered the required properties.

We know that in the problems of the ordinary Theory of Maxima and Minima it is not always necessary that a maximum

or a minimum exist.\* It is certain that every variable has an upper and a lower limit within any region for which this variable has a meaning. Therefore there exists a limit  $l$  such that all values which a variable can assume are greater than  $l$ , and that everywhere in the neighborhood of  $l$  there are values which the variable can assume. We call  $l$  the *lower* limit of the variable. In the same way there is an *upper* limit. These limits need not always be reached. There are consequently two cases possible: Either the values which are denoted as upper and lower limits may *in reality* be reached by the variables, or the variables may only come indefinitely near without ever reaching these limits. It is therefore inadmissible to presuppose the existence of a maximum or a minimum. For example, Newton's problem, cited above, has no solution, and in the case of the first problem there is sometimes a minimum and sometimes no such minimum exists.

## PROBLEMS.

It is suggested that the student select two or three of the following problems and apply the same methods of solution to them as will be done for the six problems already proposed.

1. *Problem of least action.* Find the minimum value of the integral

$$u = \int \sqrt{(x+a)(1+p^2)} \, dx,$$

the limiting values of  $x$  and  $p = \frac{dy}{dx}$  being fixed, and determine under what conditions a *parabolic arc* is in reality a solution of the problem. The problem may also be stated as follows: Determine the path of a particle for which the action  $\int v ds$  is a minimum between fixed points, if the velocity  $v$  at any point is that due to a fall from a straight line  $x+a=0$ , the axis of  $X$  being vertically downwards. [See Todhunter, *Researches in the Calculus of Variations*, p. 147; see also *Quarterly Journal of Mathematics*, Nov., 1868.] The discontinuity here is very similar to that which we shall find in the case of Problem I, p. 1.

2. *Principle of least action in the elliptic motion of a planet.* A particle is projected from a given point with a given velocity and is attracted to a fixed point by a force varying inversely as the square of the distance. Determine the path of minimum action to a second fixed point. [See Todhunter, *Researches*, etc., p. 160; Todhunter, *History of the Calculus of Variations*, p. 251; Jellett, *Calculus of Variations*, p. 76; Jacobi, *Crelle*, vol. 17, p. 68; Liouville's *Journ.*, tom. III, p. 44; Delaunay, *Liouville's Journ.*, tom. VI, p. 209.]

3. Determine the curve which renders the integral  $u = \int y x ds$  a maximum, when

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\*See *Lectures*, etc. (loc. cit.), p. 86.

the variables are given fixed limits. [See Euler, *Methodus Inveniendi Lineas Curvas Maximi Minimive Gaudentes*, Lausanne, 1794, p. 52; Woodhouse, *A Treatise on Isoperimetrical Problems and the Calculus of Variations*, p. 124.]

4. Given the length of a curve, determine its nature, when the volume generated by its rotation about a fixed axis is a maximum or a minimum. [See Euler, *Methodus*, etc., p. 196; Woodhouse, *A Treatise*, etc., p. 125; Moigno et Lindelöf, *Calcul de Variations*, p. 216; Jellett, *Calculus of Variations*, p. 160.]

5. Required the curve that, by a revolution about a fixed axis, generates the greatest or the least volume, the surface-area being constant. [See Euler, *Methodus*, etc., p. 194; Moigno et Lindelöf, *Calcul de Variations*, p. 218; Delaunay, *Liouville's Journ.*, tom. VI, p. 315; *Phil. Mag.*, 1866; Todhunter, *Researches*, etc., p. 68; Jellett, *Calculus of Variations*, p. 161 and note, p. 364.]

6. Find the curve which generates by its rotation the solid of greatest volume, the length of the curve and its area being given. [See Lacroix, *Calc. Diff'l et Int.*, Vol. II, p. 713; consult further the references above for this and the following problems.]

7. Find the curve of quickest descent when the length of the curve is given. [See John Bernoulli's Works, Vol. II, p. 255; *Mémoires de l'Académie des Sciences*, Paris, 1718, p. 120.]

8. A plane curve being given, determine a second curve of given length such that the area inclosed between the two curves be a maximum.

9. Among all curves of the same length, find the one which, by its revolution about an axis, will generate the greatest or the smallest surface-area.

10. It is required to maximize or minimize the integral

$$u = \int_{x_0}^{x_1} \phi(x, y, z) \sqrt{1 + y'^2 + z'^2} dx,$$

where  $\phi$  is a given function of the variables  $x, y, z$ , which are connected by the equation  $f(x, y, z) = 0$ ,  $f$  being a known function.

11. Find the curve of minimum length between two fixed points in space, the radius of curvature being a constant.

12. Find the form which a homogeneous body of given volume must take that its attraction upon a material point in a definite direction be as great as possible.

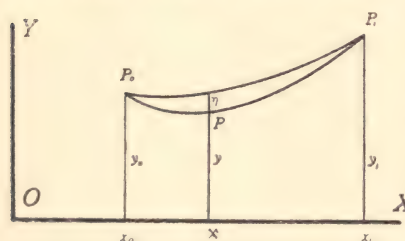
## CHAPTER II.

EXAMPLES OF SPECIAL VARIATIONS OF CURVES. APPLI-  
CATIONS TO THE CATENARY.

22. Let us consider again the integral of Art. 6,

$$\frac{S}{2\pi} = \int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad [1]$$

Suppose that there is a minimum surface-area that is generated by the rotation of a curve between the two fixed points  $P_0$  and  $P_1$  and let this curve be  $y=f(x)$ . Let  $\eta$  be the distance between



this curve and any neighboring curve measured on the  $y$ -ordinate, and suppose that  $\eta$  is a continuous function of  $x$  subject to the conditions: that for  $x=x_0$ ,  $\eta=0$ ; for  $x=x_1$ ,  $\eta=0$ ; and for all other points  $|\eta| < \rho$ , where  $\rho$  may be as small as we choose.

$$\eta' = \frac{d\eta}{dx} \quad \text{and} \quad \int_{x_0}^{x_1} \eta' dx = [\eta]_{x_0}^{x_1} = 0.$$

The integral of any neighboring curve corresponding to [1] is

$$\int_{x_0}^{x_1} (y + \eta) \sqrt{1 + \left(\frac{d(y + \eta)}{dx}\right)^2} dx. \quad [2]$$

Hence the *total variation* caused in [1] when, instead of  $y=f(x)$ , we take a neighboring curve, is

$$\frac{\Delta S}{2\pi} = \int_{x_0}^{x_1} (y+\eta) \sqrt{1+\left(\frac{d(y+\eta)}{dx}\right)^2} dx - \int_{x_0}^{x_1} y \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx. \quad [3]$$

$\Delta S$  has always a positive sign, since the surface in question is a minimum.

23. Instead of the one neighboring curve, we may consider a whole bundle of such curves, if for  $\eta$  we substitute  $\epsilon \eta$ , where  $\epsilon$  is independent of  $x$  and has any value between  $-1$  and  $+1$ . The expression [3] becomes then

$$\frac{\Delta S}{2\pi} = \int_{x_0}^{x_1} (y + \epsilon \eta) \sqrt{1+\left(\frac{d}{dx}(y + \epsilon \eta)\right)^2} dx - \int_{x_0}^{x_1} y \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx; \quad [4]$$

and, developing  $\Delta S$  by Taylor's Theorem,

$$\Delta S = \epsilon \delta S + \frac{\epsilon^2}{1.2} \delta^2 S + \frac{\epsilon^3}{1.2.3} \delta^3 S + \dots \quad [5]$$

There is no constant term in this last development, since when  $\epsilon$  is made zero in [4] the first and second integrals cancel each other.

$\delta S$  is known as the *first variation*,

$\delta^2 S$  is called the *second variation*, etc.

Instead of taking  $\eta$  a very small quantity, we may take  $\epsilon$  so small that  $\epsilon \eta$  is as small as we choose.

With Lagrange (Misc. Taur., tom. II, p. 174), writing  $\eta = \delta y$ , it is seen that the total change in  $y$  is  $\epsilon \eta = \epsilon \delta y = \Delta y$ .

REMARK. The sign of differentiation and the sign of variation may be interchanged; for example, the 1st derivative of a variation is equal to the 1st variation of a derivative, as is seen by writing

$$\eta = \delta y, \text{ then } \eta' = (\delta y)' = \frac{d}{dx} (\delta y). \quad [6]$$

Again  $\eta = \delta y$ ; change  $y$  into  $y + \epsilon \eta$ , and consequently  $y'$  into  $y' + \epsilon \eta'$ . Hence  $\eta'$  is the first variation of  $y'$ , so that

$$\eta' = \delta y' = \delta \left( \frac{dy}{dx} \right); \quad [ii]$$

and therefore from [i] and [ii]

$$\frac{d}{dx} (\delta y) = \delta \left( \frac{dy}{dx} \right).$$

It follows too that owing to the presupposed existence of  $\eta'$ , we must also assume the existence of the second differential coefficient of  $y$ .

24. Returning to [4], write  $y' = \frac{dy}{dx}$ ,  $\eta' = \frac{d\eta}{dx}$ . Then expand-

ing the expression under the sign of integration

$$(y + \epsilon \eta) \sqrt{1 + (y' + \epsilon \eta')^2} - y \sqrt{1 + y'^2},$$

we have

$$\epsilon \left\{ \eta \sqrt{1 + (y' + \epsilon \eta')^2} + \frac{(y + \epsilon \eta)(y' + \epsilon \eta')\eta'}{\sqrt{1 + (y' + \epsilon \eta')^2}} \right\}_{\epsilon=0} + \epsilon^2(\dots).$$

Hence, equating the coefficients of the 1st power of  $\epsilon$  in [4] and in [5] we have

$$\frac{\delta S}{2\pi} = \int_{x_0}^{x_1} \left( \sqrt{1 + y'^2} \cdot \eta + \frac{y y'}{\sqrt{1 + y'^2}} \eta' \right) dx,$$

which is a homogeneous function of the first degree in  $\eta$  and  $\eta'$ . The quantity  $\eta'$  cannot be indefinitely large, since then the development would not be necessarily convergent; but see Art. 116.

In a similar manner we may find a definite integral for the second variation, in which the integrand is an integral homogeneous function of the second degree in  $\eta$  and  $\eta'$ ; similarly for the third variation, etc.

25. As a form of the integrals which were given in Problems I, II, III and IV of the preceding Chapter, consider the integral

$$I = \int_{x_0}^{x_1} F(x, y, y') dx,$$

where  $F(x, y, y')$  is a known function of  $x, y$  and  $y'$ , and where the limits of this integral,  $x_1$  and  $x_0$ , are fixed. Hence, as above,

$$\begin{aligned} \Delta I &= \int_{x_0}^{x_1} F(x, y + \epsilon \eta, y' + \epsilon \eta') dx - \int_{x_0}^{x_1} F(x, y, y') dx \\ &= \int_{x_0}^{x_1} [F(x, y + \epsilon \eta, y' + \epsilon \eta') - F(x, y, y')] dx. \end{aligned}$$

This expression, when expanded by Taylor's Theorem, is

$$\Delta I = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \epsilon \eta + \frac{\partial F}{\partial y'} \epsilon \eta' + \epsilon^2 (\quad) + \dots \right) dx.$$

We also have, as in Art. 23,

$$\Delta I = \epsilon \delta I + \frac{\epsilon^2}{1.2} \delta^2 I + \dots;$$

and by comparing the coefficients of  $\epsilon$  in these two expressions, it follows that

$$\delta I = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right) dx. \quad (\text{A})$$

In the particular case given in Art. 22,  $F = y \sqrt{1 + y'^2}$ . Hence

$$\frac{\partial F}{\partial y} = \sqrt{1 + y'^2} \quad \text{and} \quad \frac{\partial F}{\partial y'} = \frac{y y'}{\sqrt{1 + y'^2}};$$

and when these relations are substituted in (A) we have, as in Art. 24,

$$\Delta I = \int_{x_0}^{x_1} \left( \sqrt{1 + y'^2} \eta + \frac{yy'}{\sqrt{1 + y'^2}} \eta' \right) dx.$$

26. From the relation

$$\Delta I = \epsilon \delta I + \frac{\epsilon^2}{1.2} \delta^2 I + \dots,$$

it is seen that when  $\epsilon$  is taken very small,  $\epsilon^2$  is as near as we wish to zero; and consequently when  $\epsilon$  is positive and indefinitely small,  $\Delta I$  is *positive*. On the other hand, when  $\epsilon$  is indefinitely small and negative,  $\Delta I$  is *negative*.

Hence the total variation  $\Delta I$  of the integral will be either positive or negative according as  $\epsilon$  is positive or negative, so long as  $\delta I$  is different from zero; and consequently there can be neither a maximum nor a minimum value of the integral.

We know, however, if  $I$  is a maximum  $\Delta I$  is always *negative*, and if  $I$  is a minimum  $\Delta I$  is always *positive*; and consequently in order to have a maximum or a minimum value of the integral,  $\delta I$  must be zero.

27. Applying the above result to the example given in Art. 22 we have

$$0 = \int_{x_0}^{x_1} \left\{ \sqrt{1 + y'^2} \eta + \frac{yy'}{\sqrt{1 + y'^2}} \frac{d\eta}{dx} \right\} dx. \quad [6]$$

Integrating by parts,

$$\int_{x_0}^{x_1} \frac{yy'}{\sqrt{1 + y'^2}} d\eta = \left[ \frac{yy'}{\sqrt{1 + y'^2}} \eta \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \left\{ \frac{yy'}{\sqrt{1 + y'^2}} \right\} \eta dx;$$

and since, by hypothesis (see Art. 22),  $\eta=0$  at both of the fixed points  $P_0$  and  $P_1$ , we have

$$\left[ \frac{yy'}{\sqrt{1+y'^2}} \eta \right]_{x_0}^{x_1} = 0.$$

Hence [6] may be written

$$0 = \int_{x_0}^{x_1} \left\{ \sqrt{1+y'^2} - \frac{d}{dx} \left( \frac{yy'}{\sqrt{1+y'^2}} \right) \right\} \eta \, dx. \quad [7]$$

28. We assert that in the expression above

$$\sqrt{1+y'^2} - \frac{d}{dx} \left\{ \frac{yy'}{\sqrt{1+y'^2}} \right\}$$

must always be zero between the limits  $x_0$  and  $x_1$ . For, assuming that the contrary is the case; then, since  $\eta$  is arbitrary, we may, with Heine,\* write

$$\eta = (x-x_0)(x_1-x) \left\{ \sqrt{1+y'^2} - \frac{d}{dx} \left( \frac{yy'}{\sqrt{1+y'^2}} \right) \right\},$$

where  $\eta$  becomes zero for the valued  $x=x_0$  and  $x=x_1$ . Substituting this value of  $\eta$  in [7], we have

$$= \int_{x_0}^{x_1} \left\{ \sqrt{1+y'^2} - \frac{d}{dx} \left[ \frac{yy'}{\sqrt{1+y'^2}} \right] \right\}^2 (x-x_0)(x_1-x) \, dx, \quad [8]$$

an expression which is positive within the whole interval  $x_0 \dots x_1$ .

The integrand in [8], looked upon as a sum of infinitely small elements, has all its elements of the same sign and positive; so that the only possible way for the right-hand member of [8] to be zero is that

$$\sqrt{1+y'^2} - \frac{d}{dx} \left[ \frac{yy'}{\sqrt{1+y'^2}} \right] = 0.$$

\* Heine, Crelle's Journal, bd. 54, p. 338.

We therefore have a differential equation of the second order for the determination of the unknown quantity  $y$ .

29. This differential equation is a special case of the more general differential equation, which may be derived from the integral

$$I = \int_{x_0}^{x_1} F(y, y') dx;$$

whence, as before (Arts. 25 and 27),

$$\delta I = \int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right\} dx = \int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right\} \eta dx.$$

As in Art. 27, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left[ \frac{\partial F}{\partial y'} \right] = 0,$$

or

$$\frac{\partial F}{\partial y} = \frac{d}{dx} \left[ \frac{\partial F}{\partial y'} \right]. \quad [9]$$

But

$$dF(y, y') = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy', \quad [10]$$

or

$$dF(y, y') - \left\{ \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy' \right\} = 0.$$

Hence from [9],

$$dF(y, y') - \left\{ \frac{d}{dx} \left[ \frac{\partial F}{\partial y'} \right] dy + \frac{\partial F}{\partial y'} dy' \right\} = 0,$$

or

$$dF(y, y') - d \left\{ y' \frac{\partial F}{\partial y'} \right\} = 0,$$

and integrating,

$$F(y, y') - y' \frac{\partial F}{\partial y'} = C, \quad [11]$$

where  $C$  is the constant of integration.

The relation [11] exists only when the integrand of the given integral does not contain *explicitly* the variable  $x$ ; otherwise the relation [10] would not be true, and then we could not deduce [11].

30. Applying this relation [11] to the special case above (Art. 28) where

$$F(y, y') = y \sqrt{1 + y'^2},$$

we have

$$y \sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} = m,$$

$m$  being the constant of integration, a quantity which will be considered more in detail later.

The above expression may be written

$$\frac{y(1 + y'^2 - y'^2)}{\sqrt{1 + y'^2}} = m,$$

or

$$y = m \sqrt{1 + y'^2}. \quad [\text{I}]$$

From [I] it follows directly that

$$y^2 - m^2 = m^2 \left( \frac{dy}{dx} \right)^2; \quad [\text{II}]$$

and [II], differentiated with respect to  $x$ , is

$$y = m^2 \frac{d^2y}{dx^2}.$$

Two solutions of this differential equation are

$$y = e^{x/m} \quad \text{and} \quad y = e^{-x/m},$$

so that the general solution is

$$y = c_1 e^{x/m} + c_2 e^{-x/m}. \quad [\text{III}]$$

It appears that we have in this expression three arbitrary constants,  $m$ ,  $c_1$ , and  $c_2$ ; but from [II] we have, after substituting for  $y^2$  and  $\left(\frac{dy}{dx}\right)^2$  their values from [III],

$$m^2 = 4 c_1 c_2.$$

Hence, writing in [III],

$$c_1 = \frac{1}{2}m e^{-x_0'/m} \text{ and } c_2 = \frac{1}{2}m e^{x_0'/m},$$

where  $x_0'$  is a constant, we have

$$y = \frac{1}{2}m [e^{(x-x_0')/m} + e^{-(x-x_0')/m}]. \quad [\text{III}']$$

The two constants  $x_0'$  and  $m$  are determined from the two conditions that the curve is to pass through the two fixed points  $P_0$  and  $P_1$ .

31. From what was given in Art. 19 it would appear that two neighboring curves are distinct throughout at least certain portions of their extent. This implies the existence of a certain *neighborhood* about the curve  $C$  that is supposed to offer a minimum, within which this curve is not intersected by a neighboring curve. Suppose that the curve  $C_\epsilon$  is derived from the curve  $C$  by the substitution of  $y + \epsilon \eta$  for  $y$  (cf. Art. 22). Consider the family of curves  $(C_\epsilon)$  obtained by varying  $\epsilon$  between  $-1$  and  $+1$ . For sufficiently small values of  $\epsilon$  the curve  $C_\epsilon$  will lie within the neighborhood presupposed to exist, and a portion of our family of curves will lie within this neighborhood. This is a *necessary consequence* of the supposed existence of a minimal surface of revolution. As a condition, however, it is not sufficient to assure the existence of a curve giving such a surface. The fact that the surfaces generated by the curves  $C_\epsilon$  are all greater than that generated by the curve  $C$  does not prevent the existence of a neighboring curve constructed after a manner other than that by which the curves  $C_\epsilon$  are produced, which would generate a surface of revolution having less surface-area than that due to the revolution of  $C$ .

It is useful to determine for just what curve  $C$  the above condition may be satisfied, and while this does not prove that the curve  $C$  gives a minimal surface of revolution, it will at least limit the range of curves among which we may hope to find a generator of a minimal surface. Further investigation of this more limited range of curves may locate the curve or curves giving a minimal surface, if such exists, and in the other case may prove their non-existence. In the further investigation we shall derive the sufficient conditions to assure the existence of a maximum or a minimum.

32. The conclusions drawn from Art. 30 show that, if a curve exists which offers the required minimal surface, that curve must be a catenary. Since the catenary must pass through the two fixed points  $P_0$  and  $P_1$ , we may determine the constants  $m$  and  $x_0'$  from the two relations (see formula [III'], Art. 30):

$$y_0 = \frac{1}{2}m [e^{(x_0-x_0')/m} + e^{-(x_0-x_0')/m}],$$

$$y_1 = \frac{1}{2}m [e^{(x_1-x_0')/m} + e^{-(x_1-x_0')/m}].$$

We shall see in the next Chapter that three cases arise according as the solution of the above equations furnish us with *two* catenaries, *one* catenary, or *no* catenary.

In the first place, it may be shown that the catenary nearest the  $X$ -axis can never furnish a minimal surface. The second case arises from the coincidence of the two catenaries just mentioned, and it will be seen that an infinite number of curves may in this case be drawn between the two points, each of which gives rise to the same rotation-area. These results are due to Todhunter (see references at the beginning of the next Chapter).

## CHAPTER III.

## PROPERTIES OF THE CATENARY.

33. Owing to certain theorems that have been discovered by Lindelöf and other writers, some of the very characteristics of a minimal surface of rotation, which are sought in the Calculus of Variations, may be obtained for the case of the revolution of the catenary without the use of that theory. We shall give these results here, as they offer a handy method of comparison when we come to the results that have been derived through the methods of the Calculus of Variations.

In presenting the subject-matter of this Chapter, the lectures given by Prof. Schwarz at Berlin are followed rather closely. The results are derived by Todhunter in a somewhat different form in his *Researches in the Calculus of Variations*, p. 54; see also the prize essay of Goldschmidt, *Monthly Notices of the Royal Astronomical Society*, Vol. 12, p. 84; Jellett, *Calculus of Variations*, 1850, p. 145; Moigno et Lindelöf, *Calcul des Variations*, 1861, p. 204; etc.

34. Take the equation of the catenary which was given in the preceding Chapter, Art. 30, in the form\*

$$y = \frac{1}{2}m[e^{(x-x_0')/m} + e^{-(x-x_0')/m}].$$

It follows at once that

$$m \frac{dy}{dx} = \pm \sqrt{y^2 - m^2} = \frac{1}{2}m[e^{(x-x_0')/m} - e^{-(x-x_0')/m}].$$

On the right-hand side of the equation stands a one-valued function, but on the left-hand side, a two-valued function. It is therefore necessary to define the left-hand side so that it will be a one-valued function corresponding to the right-hand side.

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\*Throughout this discussion the  $X$ -axis is taken as the directrix.

If we make  $x > x_0'$ , then is

$$e^{(x-x_0')/m} > e^{-(x-x_0')/m},$$

and consequently  $\sqrt{y^2 - m^2}$  is positive. But when  $x < x_0'$ , it is seen that

$$e^{(x-x_0')/m} < e^{-(x-x_0')/m},$$

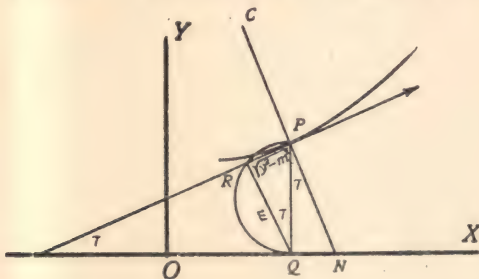
and then  $\sqrt{y^2 - m^2}$  is negative. It therefore follows that there is only one root of  $\frac{dy}{dx} = 0$ , and this is for the value  $x = x_0'$ . The corresponding value of  $y$  is  $m$ .

This value  $m$  is the smallest value that  $y$  can have; for  $\frac{dy}{dx} = 0$  is the condition for a maximum or a minimum value, and since  $\frac{d^2y}{dx^2}$  is positive for  $x = x_0'$ , it follows that  $m$  is a minimum value of  $y$ . Further, since  $\sqrt{y^2 - m^2}$  is continuously positive or continuously negative, there is no maximum value of  $y$ . The tangent to the curve at the point  $x = x_0'$ ,  $y = m$  is parallel to the  $X$ -axis, since at this point  $\frac{dy}{dx} = 0$ .

35. At every point of the curve we have

$$\frac{dy}{dx} = \tan \tau = \frac{\sqrt{y^2 - m^2}}{m}.$$

Hence, to construct a tangent at any point of the catenary, for ex-



ample at  $P$ , drop the perpendicular  $PQ$ , and describe the semi-circle on  $PQ$  as diameter. Then, with radius equal to  $m$ , draw a circle from  $Q$  as center, which cuts the semi-circle at  $R$ ; join  $R$  and  $P$ . The line  $RP$  is the required tangent.

$$\text{Again } ds^2 = dx^2 + dy^2 = \left\{ 1 + \frac{y^2 - m^2}{m^2} \right\} dx^2 = \frac{y^2}{m^2} dx^2;$$

consequently

$$ds = \frac{y dx}{m} = \frac{1}{2} [e^{(x-x_0')/m} + e^{-(x-x_0')/m}] dx;$$

and integrating,

$$s - s_0' = \frac{1}{2} m \left[ e^{(x-x_0')/m} - e^{-(x-x_0')/m} \right] = \sqrt{y^2 - m^2},$$

where  $s_0'$  denotes that the arc is measured from the lowest point of the catenary.

The geometrical locus of  $R$  is a curve which cuts all the tangents to the catenary at right angles, and is therefore the *orthogonal trajectory* of this system of tangents. This trajectory has the remarkable property that the perpendiculars  $QR$ , etc., of length  $m$ , which are employed in the construction of the tangents to the catenary, are themselves tangent to the trajectory.

This trajectory possesses also the remarkable property that, if we rotate it around the  $X$ -axis, the surface of rotation has a constant curvature,

Further,  $PN$ , the normal to the catenary,

$$= y \sec \tau = \frac{y^2}{m}, \text{ and}$$

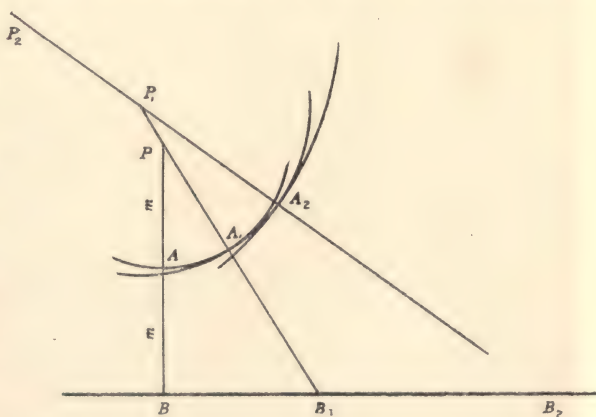
$$\rho = \frac{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left( \frac{ds}{dx} \right)^3}{\frac{d^2y}{dx^2}} = \frac{\left( \frac{y}{m} \right)^3}{\frac{y}{m^2}} = \frac{y^2}{m},$$

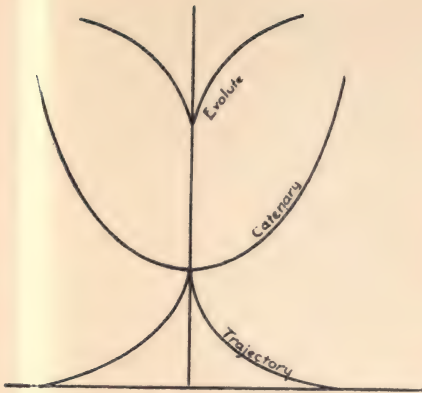
or

$$PN = PC \text{ (see figure),}$$

where  $PC$  is the length of the radius of curvature.

36. *The geometrical construction of the catenary.* Take an ordinate equal to  $2m$ . This determines the point  $P$  (see figure). With  $P$  as center and radius equal to  $m$ , describe a circle. This intersects  $PB$  at a point  $A$ , say. On the circumference of this circle take a point  $A_1$ , very near  $A$ , and draw the line  $PA_1B_1$ , and on this line extended take  $P_1$  such that  $P_1A_1 = A_1B_1$ . With radius  $P_1A_1$  draw another circle, and on





this circle take a point  $A_2$ , very near the point  $A_1$ , and draw the line  $P_1A_2B_2$ . Take on this line extended the point  $P_2$  so that  $P_2A_2=A_2B_2$ , etc. The locus of the points  $A$  is the required catenary.

The accompanying figure shows approximately the relative positions of the catenary, its evolute and the trajectory.

37. It appears from the previous article that a catenary is completely determined when we know any point on it and the tangent at this point. This may be proved analytically as follows:

Let  $\bar{x}, \bar{y}$  be a point through which passes a straight line, making with the  $X$ -axis an angle whose tangent is  $k$ . The conditions that a catenary pass through this point and have the given line as tangent are:

$$\bar{y} = \frac{m}{2} [e^{(\bar{x}-x_0')/m} + e^{-(\bar{x}-x_0')/m}],$$

$$k = \bar{y}' = \frac{1}{2} [e^{(\bar{x}-x_0')/m} - e^{-(\bar{x}-x_0')/m}].$$

For brevity write  $e^{(\bar{x}-x_0')/m} = z$ , so that the above conditions become

$$\bar{y} = \frac{m}{2} (z + z^{-1}), \quad k = \frac{1}{2} (z - z^{-1}).$$

Hence,

$$z^2 - 2kz - 1 = 0;$$

therefore

$$z = k \pm \sqrt{1 + k^2}$$

and

$$z^{-1} = -k \pm \sqrt{1 + k^2}.$$

We therefore have

$$\bar{y} = \pm m \sqrt{1 + k^2}.$$

Since  $\bar{y}$  and  $m$  are both positive, it follows that we may take only the upper sign. Consequently, if we write

$$k = \tan \alpha,$$

we have

$$z = \tan a + \sqrt{1 + \tan^2 a} = \frac{\sin a + 1}{\cos a},$$

$$-z^{-1} = \tan a - \sqrt{1 + \tan^2 a} = \frac{\sin a - 1}{\cos a},$$

and

$$m = \frac{\bar{y}}{\sqrt{1 + \tan^2 a}} = \bar{y} \cos a.$$

Further, since  $\log z$  has one and only one real value for a definite value of  $z$ , the constant  $x_0'$  is determined uniquely from

$$\frac{x - x_0'}{m} = \log z = \log \frac{\sin a + 1}{\cos a},$$

and the quantities  $x_0'$  and  $m$  determine uniquely a catenary which has the given line as tangent at the point  $x, y$ .

38. In particular, consider the catenary that has the  $Y$ -axis as the axis of symmetry, and let the two points  $P_0$  and  $P_1$  be at equal heights on the curve so that their coordinates are, say  $(-a, b)$  and  $(a, b)$ .

The equation of the catenary is now, since  $x_0' = 0$ ,

$$y = \frac{m}{2} (e^{x/m} + e^{-x/m});$$

and consequently

$$b = \frac{m}{2} (e^{a/m} + e^{-a/m}) = \phi(m), \text{ say,} \quad [1]$$

where we regard  $a$  as constant and  $m$  variable.

We wish to determine whether this last equation gives a real value or real values for  $m$ . We see that  $\phi(m)$  is infinite when  $m=0$  and also when  $m=\infty$ .

Further

$$2 \phi'(m) = e^{a/m} + e^{-a/m} - \frac{a}{m} (e^{a/m} - e^{-a/m}),$$

or

$$\phi'(m) = 1 - \frac{1}{2} \frac{a^2}{m^2} - \frac{3}{4!} \frac{a^4}{m^4} - \dots - \frac{2n-1}{(2n)!} \frac{a^{2n}}{m^{2n}} - \dots,$$

so that  $\phi'(m)$  is negative *infinity* when  $m$  is zero; is *unity* when  $m$  is infinite, and changes sign once and only once as  $m$  passes

from zero to infinity. The least value that  $\phi(m)$  can have is for the value of  $m$  that satisfies  $\phi'(m)=0$ .

If, then, the given value of  $b$  is greater than the least value of  $\phi(m)$ , there are two values of  $m$  which satisfy [1]; if the given value of  $b$  be equal to the least value of  $\phi(m)$ , there is only one value of  $m$ ; and if the given value of  $b$  is less than the least value of  $\phi(m)$ , there is no possible value of  $m$ .

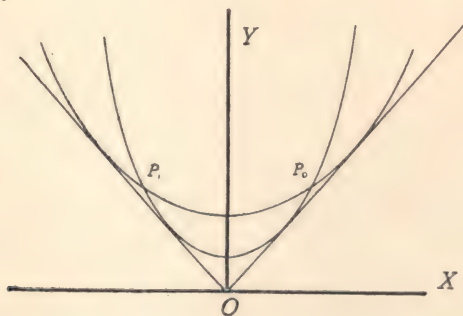
Moigno and Lindelöf have shown that the value of  $\frac{a}{m}$  which satisfies

$$e^{a/m} + e^{-a/m} - \frac{a}{m}(e^{a/m} - e^{-a/m}) = 0$$

is approximately  $\frac{a}{m} = 1.19968\dots$ ; and then from [1] it follows that  $\frac{b}{m} = 1.81017\dots$ ; and therefore  $\frac{b}{a} = 1.50888\dots = \tan(56^\circ 28')$  approximately (see Todhunter, loc. cit., Art. 60). Thus there are *two* catenaries satisfying the prescribed conditions, or *one* or *none*, according as  $\frac{b}{a}$  is greater than, equal to, or less than 1.50888...

If we write  $k = \frac{b}{a} = \tan(56^\circ 28')$ , it is seen that  $y = kx$  and  $y = -kx$  are the two tangents to the catenary that may be drawn through the origin.

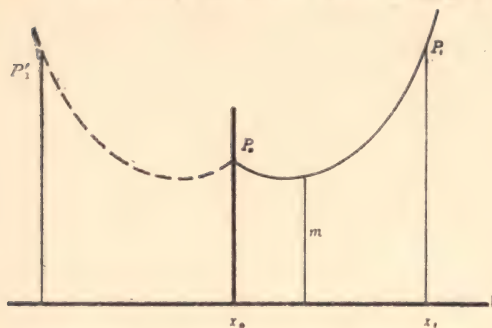
As the ratio  $b/a$  is independent of  $m$ , it also follows that all the catenaries of the form  $y = m/2 (e^{x/m} + e^{-x/m})$ , which may be derived by varying  $m$ , have the same two tangent lines through the origin, the points of contact being  $x = \pm 1.19968\dots m$  and  $y = 1.81017\dots m$ .



39. Returning to the catenary  $y = \frac{1}{2}m[e^{(x-x_0')/m} + e^{-(x-x_0')/m}]$ , we shall see that also here there are three cases which come under investigation according as:

- I. Two catenaries may be drawn through the fixed points;
- II. One catenary may be drawn through these points;
- III. No catenary may be drawn through the two points.

We may assume that  $y_1 \geq y_0$ ,  $x_1 > x_0$ . For if  $x_1 < x_0$ , we would



only have to change the direction of the  $X$ -axis which we name positive and negative; or we might consider the case of  $P_0$  and  $P'_1$ , where  $P'_1$  is the image of  $P_1$ ; that is, the point symmetrically situated to  $P_1$  on the other side of the  $y_0$ -ordinate.

40. From the equation of the catenary it follows that

$$y_0 = \frac{1}{2}m [e^{(x_0 - x'_0)/m} + e^{-(x_0 - x'_0)/m}],$$

and

$$y_0^2 - m^2 = \frac{1}{4}m^2 [e^{(x_0 - x'_0)/m} - e^{-(x_0 - x'_0)/m}]^2.$$

Therefore

$$\sqrt{y_0^2 - m^2} = \pm \frac{1}{2}m [e^{(x_0 - x'_0)/m} - e^{-(x_0 - x'_0)/m}]; \quad [I]$$

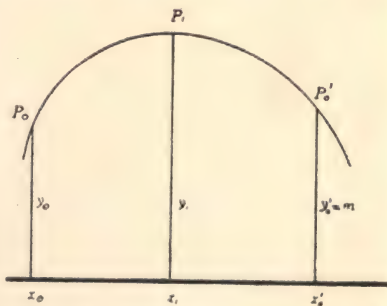
and from this relation it is seen that  $\sqrt{y_0^2 - m^2}$  has a positive or negative sign according as  $x_0 - x'_0 \gtrless 0$ . Hence, also,

$$(x_0 - x'_0)/m = \pm \log \text{nat} [(y_0 + \sqrt{y_0^2 - m^2})/m]. \quad [a]$$

41. Under the assumption that  $y_1 \geq y_0$ , we must first show that such a figure as the one which follows cannot exist in the present discussion. We know that

$$y_1 = \frac{1}{2}m [e^{(x_1 - x'_0)/m} + e^{-(x_1 - x'_0)/m}].$$

That  $x_1 - x'_0$  is necessarily positive is seen from the fact that the ordinate  $y'_0 = m$  corresponds to the value  $x'_0$ , and is a minimum. (See Art. 34.) Suppose that  $x'_0 > x_1$ . By hypothesis  $y_1 \geq y_0$ , and further  $m \leq y_0$ , and consequently  $m \leq y_1$ . The form of the curve is then that given in the figure; and we have within the interval  $x_0$  to  $x'_0$  a value of  $x$ , for which the ordinate  $y$  is greater than it is at the end-points.  $y$



must therefore have within this interval a maximum value. But we have shown (Art. 34) that there is no maximum value\* of  $y$ ; hence,

$$\sqrt{y_1^2 - m^2} = +\frac{1}{2}m[e^{(x_1-x_0')/m} - e^{-(x_1-x_0')/m}],$$

and there cannot be the *minus* sign as in equation [I]; hence,

$$(x_1 - x_0')/m = +\log \text{nat} [(y_1 + \sqrt{y_1^2 - m^2})/m]. \quad [b]$$

42. Eliminate  $x_0'$  from [a] and [b] and noting that in [a] there is the  $\pm$  sign, we have two different functions of  $m$ , which may be written:

$$\begin{aligned} f_1(m) &= \log \text{nat} [(y_1 + \sqrt{y_1^2 - m^2})/m] \\ &\quad - \log \text{nat} [(y_0 + \sqrt{y_0^2 - m^2})/m] - (x_1 - x_0)/m, \end{aligned}$$

and

$$\begin{aligned} f_2(m) &= \log \text{nat} [(y_1 + \sqrt{y_1^2 - m^2})/m] \\ &\quad + \log \text{nat} [(y_0 + \sqrt{y_0^2 - m^2})/m] - (x_1 - x_0)/m, \end{aligned}$$

two functions of a transcendental nature, which we have now to consider. We must see whether  $f_1(m)=0$ ,  $f_2(m)=0$  have roots with regard to  $m$ ; that is, whether it is possible to give to  $m$  positive real values, so that the equations  $f_1(m)=0$ ,  $f_2(m)=0$  will be satisfied. If it is possible thus to determine  $m$ , we must then see whether the values  $x_0'$  which may be derived from equations [a] and [b] are one-valued.

The first derivative of  $f_1(m)$  is

$$f_1'(m) = \frac{1}{m} \left[ \frac{y_0}{\sqrt{y_0^2 - m^2}} - \frac{y_1}{\sqrt{y_1^2 - m^2}} + \frac{x_1 - x_0}{m} \right]. \quad [c]$$

On the right-hand side of this expression  $1/m$  is positive, also  $(x_1 - x_0)/m$  is positive, and

$$\frac{1}{\sqrt{1 - m^2/y_0^2}} - \frac{1}{\sqrt{1 - m^2/y_1^2}} \text{ is positive, if } y_1 > y_0.$$

Hence  $f_1'(m)$  is positive in the interval  $0 \dots y_0$ .

\* In other words,  $y_1$  cannot be greater than  $y_0$  and at the same time  $x_0'$  greater than  $x_1$ .

$$\begin{aligned}\text{Further, } f_1(o) &= \log \text{ nat } 2y_1 - \log \text{ nat } (m=o) \\ &\quad - \log \text{ nat } 2y_0 + \log \text{ nat } (m=o) \\ &\quad - [(x_1 - x_0)/m]_{(m=o)} = -\infty.\end{aligned}$$

43. It is further seen that  $f_1(m)$  continuously increases within the interval  $o \dots y_0$ , so that  $-\infty$  is the least value that  $f_1(m)$  can take.

Again

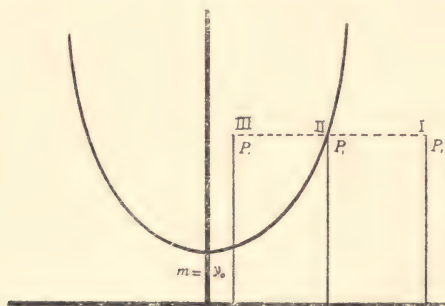
$$f_1(y_0) = \log \text{ nat } \left[ \frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0} \right] - \frac{x_1 - x_0}{y_0}. \quad [\text{II}]$$

Then if

- I.  $f_1(y_0) < o$ ,  $f_1(m)$  has no root;
- II.  $f_1(y_0) = o$ ,  $f_1(m)$  has one root,  $m = y_0$ ;
- III.  $f_1(y_0) > o$ ,  $f_1(m)$  has a root,  $m_1 < y_0$ .

When

$f_1(y_0) < o$ ,  $P_1$  is outside of the catenary;  
 $f_1(y_0) = o$ ,  $P_1$  is on the catenary;  
 $f_1(y_0) > o$ ,  $P_1$  is within the catenary.



This may be show as follows :

$$y = \frac{1}{2} y_0 [e^{(x-x_0)/y_0} + e^{-(x-x_0)/y_0}];$$

since when  $y = m$ ,  $x = x_0'$ ; and, therefore, when  $y = y_0 = m$ ,  $x = x_0$ .

We also have

$$y^2 - y_0^2 = \frac{1}{4} y_0^2 [e^{(x-x_0)/y_0} - e^{-(x-x_0)/y_0}]^2.$$

Hence

$$\sqrt{y^2 - y_0^2} = \pm \frac{1}{2} y_0 [e^{(x-x_0)/y_0} - e^{-(x-x_0)/y_0}];$$

where the positive sign is to be taken, when  $x > x_0$ , and the negative sign, when  $x < x_0$ .

We also have  $x - x_0 = y_0 \log \text{nat} [(y + \sqrt{y^2 - y_0^2})/y_0]$ . Comparing this equation with equation [II] above, and noticing the figure, it is seen that, when

$x_1 - x_0 = y_0 \log \text{nat} [(y_1 + \sqrt{y_1^2 - y_0^2})/y_0]$ , then  $P_1$  is on the catenary,

$x_1 - x_0 > y_0 \log \text{nat} [(y_1 + \sqrt{y_1^2 - y_0^2})/y_0]$ , then  $P_1$  is outside the catenary,

$x_1 - x_0 < y_0 \log \text{nat} [(y_1 + \sqrt{y_1^2 - y_0^2})/y_0]$ , then  $P_1$  is within the catenary.

Hence, when  $f_1(y) > 0$ , there is one and only one real root in the interval  $0 \dots y_0$ , and we can draw through the points  $P_1$  and  $P_0$  a catenary, for which the abscissa of the lowest point is  $< x_0$ .

44. *The discussion of  $f_2(m)$ .* We saw (Art. 42) that

$$f_2(m) = \log [(y_1 + \sqrt{y_1^2 - m^2})/m] + \log [(y_0 + \sqrt{y_0^2 - m^2})/m] - (x_1 - x_0)/m.$$

Therefore

$$f_2'(m) = -\frac{1}{m^2} \left[ \frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0) \right].$$

When  $m$  changes from  $0$  to  $y_0$ , the quantity  $\sqrt{y_0^2/m^2 - 1}$  continuously decreases, and consequently  $\frac{y_0}{\sqrt{y_0^2/m^2 - 1}}$  becomes greater

and greater. Hence if the expression  $-m^2 f_2'(m)$  takes the value  $0$ , it takes it only once in the interval from  $0$  to  $y_0$ . That this expression does take the value  $0$  within this interval is seen from the fact that, for  $m=0$ ,  $-m^2 f_2'(m) = -(x_1 - x_0)$ , where  $x_1 - x_0 > 0$ , so that  $-m^2 f_2'(m)$  has a negative value; but, for  $m=y_0$ ,  $-m^2 f_2'(y_0) = +\infty$ , so that the expression must take the value zero between these two values of  $m$ .

Let  $\mu$  be this value of  $m$  which satisfies the equation, so that

$$\frac{y_1 \mu}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0 \mu}{\sqrt{y_0^2 - \mu^2}} - (x_1 - x_0) = 0, \quad [A]$$

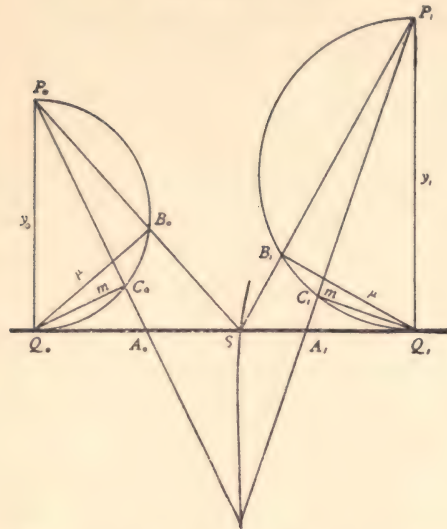
which is an algebraical equation of the eight degree in  $\mu$ , or an algebraical equation of the fourth degree in  $\mu^2$ .

45. *An approximate geometrical construction for the root that lies between 0 and  $y_0$ .* In the figure it is seen that the triangles  $P_0 Q_0 A_0$  and  $P_0 Q_0 C_0$  are similar, as are also the triangles  $P_1 Q_1 A_1$  and  $P_1 Q_1 C_1$ ; hence, if  $m$  is the length of the line  $Q_0 C_0 = Q_1 C_1$ , we have

$$Q_0 A_0 = \frac{y_0 m}{\sqrt{y_0^2 - m^2}},$$

and

$$Q_1 A_1 = \frac{y_1 m}{\sqrt{y_1^2 - m^2}}.$$



By taking equal lengths  $Q_0 C_0 = Q_1 C_1$  on the two semi-circles and prolonging  $P_0 C_0$  and  $P_1 C_1$  until they intersect, we have as the locus of the intersections a certain curve. This curve must intersect the  $X$ -axis in a point  $S$ , say. Noting that

$$Q_0 S + Q_1 S = Q_0 Q_1 = x_1 - x_0,$$

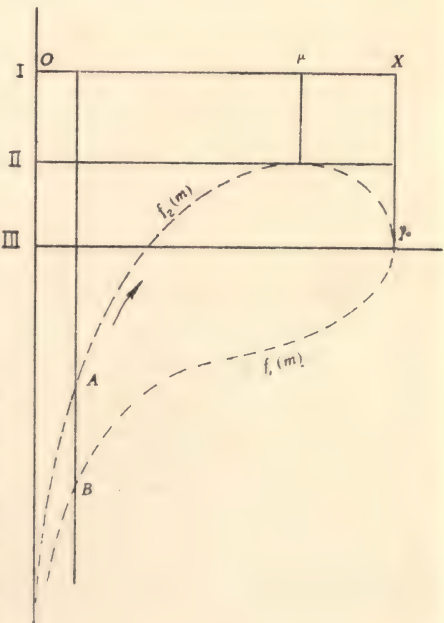
it follows that

$$\frac{y_0 \cdot Q_0 B_0}{\sqrt{y_0^2 + Q_0 B_0^2}} + \frac{y_1 \cdot Q_1 B_1}{\sqrt{y_1^2 - Q_1 B_1^2}} = x_1 - x_0,$$

which, compared with the equation [A] above, shows that

$$Q_0 B_0 = Q_1 B_1 = \mu.$$

46. *Graphical representation of the functions  $f_1(m)$  and  $f_2(m)$ .* The lengths  $m$  are measured on the  $X$ -axis. Equation [c] gives  $f_1'(y_0) = \infty$ ; that is, the tangent to the curve  $y = f_1(x)$  at the point  $y_0$  is parallel to the axis of  $y$ . Fur-



ther,  $f_1(o) = -\infty$ , so that the negative half of the axis of  $y$  is asymptotic to the curve  $y = f_1(x)$ . The branch of the curve is here algebraic, since  $y = f_1(x)$ , for  $x = o$ , is algebraically infinite.

47. Consider next the curve  $y = f_2(m)$ . It is seen that  $f_1(y_0) = f_2(y_0)$ ; and also  $f_2'(y_0) = -\infty$ , so that the tangent at this point\* is also parallel to the axis of the  $y$ . Further, the negative half of the axis of the  $y$  is an asymptote to the curve; but the branch of the curve  $y = f_2(m)$  is transcendental at the point  $m = o$ ; because logarithms enter in the development of this function in the neighborhood of  $m = o$ , as may be seen as follows:

$$f_2(m) = \log [(y_1 + \sqrt{y_1^2 - m^2})/m] + \log [(y_0 + \sqrt{y_0^2 - m^2})/m] - [(x_1 - x_0)/m] = -[(x_1 - x_0)/m] - 2 \log m + P(m),$$

where  $P(m)$  denotes a power series in positive and integral ascending powers of  $m$ ; hence, the function behaves in the neighborhood of  $m = o$  as a logarithm.

48. We saw that

$$f_2'(m) = -\frac{1}{m^2} \left[ \frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0) \right].$$

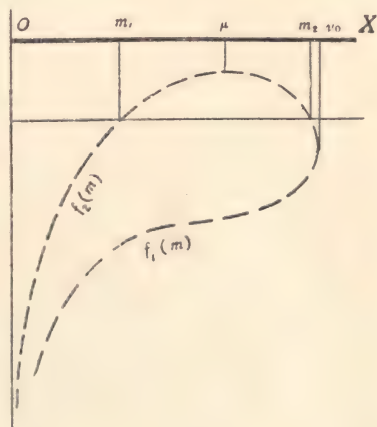
For the value  $m = \mu$  the expression within the brackets is zero, and when  $m = o$ , this expression becomes  $-(x_1 - x_0)$ , and is negative. As seen above in the interval  $m = o$  to  $m = y_0$ , the expression

$$-\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)$$

becomes greater and greater, so that between the value  $m = o$  and  $m = \mu$ , it is negative.

Furthermore,  $f_2'(m)$  is positive between  $m = o$  and  $m = \mu$ , and negative between  $m = \mu$  and  $m = y_0$ .

Hence  $f_2(m)$  increases between  $m = o$  and  $m = \mu$ , and decreases between  $m = \mu$  and  $m = y_0$ ; and consequently  $f_2(\mu)$  is a maximum.



\* The distance  $y_0$  is, of course, measured on the  $X$ -axis.

49. We must consider the function  $f_2(m)$  when  $m$  is given different values and see how many catenaries may be laid between the points  $P_0$  and  $P_1$ .

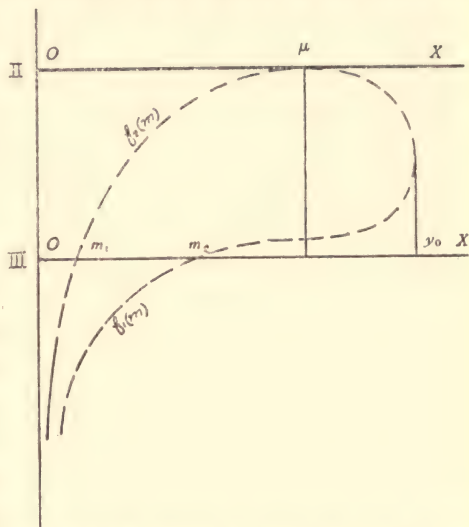
We have:

CASE I.  $f_2(\mu) < 0$ .

In this case  $f_2(m)$  is nowhere zero, and there is no root of  $f_2(m)$  which we can use. There is also no root of  $f_1(m)$ , since  $f_2(y_0) < 0$  and  $f_2(y_0) = f_1(y_0)$ , so that  $f_1(y_0) < 0$ , and there is no root (see Art. 43).

CASE II.  $f_2(\mu) = 0$ .

All values of  $m$  other than  $\mu$  cause  $f_2(m)$  to be negative, so that there is a root and only one root of the equation  $f_2(m) = 0$ , and consequently only one catenary. In this case  $f_1(m)$  can never be zero; since  $f_2(y_0) < 0$ , and  $f_1(y_0) = f_2(y_0)$ , so that  $f_1(y_0) < 0$ , with the result similar to that in Case I.



CASE III.  $f_2(\mu) > 0$ .

We have here two catenaries. One root of  $f_2(m) = 0$  lies between  $0$  and  $\mu$ , and often another between  $\mu$  and  $y_0$ , as is seen from what follows:

$$f_2(+0) = -\infty \quad \text{and} \quad f_2(\mu) > 0.$$

Since  $f_2(m)$  continuously increases in the interval  $+0 \dots \mu$ , it can take the value  $0$  only once within this interval.

In the interval  $\mu \dots y_0$ ,  $f_2(m)$  continuously decreases, so that if  $f_2(y_0) > 0$ , there is no root of  $f_2(m) = 0$  within this interval; but if  $f_2(y_0) \leq 0$ , then there is one and only one root within this interval, and in the latter case there are two catenaries.

We must next consider the roots of  $f_1(m)$ . When  $f_2(y_0) < 0$ , then is  $f_1(y_0) < 0$ , so that there is no root of  $f_1(m) = 0$ . But when  $f_2(y_0) = 0$ , then  $f_1(y_0) = 0$ ; and  $f_1(m) = 0$  has the root  $m = y_0$ , which was just considered.

Therefore:

- A*) { When  $f_2(y_0) < 0$ ,  $f_2(m)$  has two roots; and when  $f_2(y_0) = 0$ ,  
 $f_2(m)$  has a root in addition to the root which belongs to  
 $f_2(y_0) = f_1(y_0)$ .  
*B*) { But when  $f_2(y_0) > 0$ , then there is only one root for  $f_2(m) = 0$ ,  
 which lies between  $0 \dots \mu$ ; this root is denoted by  $m_1$ .

50. From the formulæ (Art. 42) for  $f_1(m)$  and  $f_2(m)$  we have:

$$f_2(m) = f_1(m) + 2 \log [(y_0 + \sqrt{y_0^2 - m^2})/m].$$

We consider the values of  $m$  within the interval  $0 \dots y_0$ ; for  $m = 0$ ,  $(y_0 + \sqrt{y_0^2 - m^2})/m = \infty$ ; and for  $m = y_0$ ,  $(y_0 + \sqrt{y_0^2 - m^2})/m = 1$ . Consequently, within this interval  $\log [(y_0 + \sqrt{y_0^2 - m^2})/m]$  is positive, and therefore also  $f_2(m) > f_1(m)$ ; and since  $f_2(m_1) = 0$ , it follows that  $f_1(m_1) < 0$ .

On the other hand,  $f_1(y_0) = f_2(y_0)$ ; and since  $f_2(y_0) > 0$ , we have  $f_1(y_0) > 0$ . Moreover, within the interval  $0 \dots y_0$ ,  $f_1(m)$  continuously increases, and  $f_1(+0) < 0$ , so that within the interval  $0 \dots m_1$ ,  $f_1(m)$  has no root, and within the interval  $m_1 \dots y_0$ , one root.

Hence, under *B*),  $f_2(m)$  has a root  $m_1$  within the interval  $0 \dots \mu$ , and only one root, and  $f_1(m)$  has a root between  $m_1$  and  $y_0$ , and only one, making a total under the heading *B*) of two catenaries.

We have the following summary:

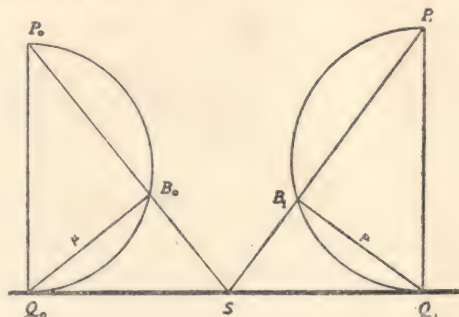
- 1°.  $f_2(\mu) < 0$ , no catenary;
- 2°.  $f_2(\mu) = 0$ , one catenary;
- 3°.  $f_2(\mu) > 0$ , two catenaries.

51. On the consideration of the intersection of the tangents drawn to the catenary at the points  $P_0$  and  $P_1$ .

CASE I. As shown above, there is no catenary, so that the consideration of the tangents is without interest.

CASE II.  $f_2(\mu) = 0$ .

Here the catenary enjoys the remarkable property that the tangents drawn at the points  $P_0$

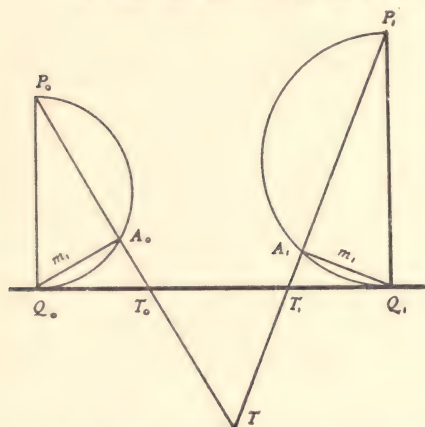


and  $P_1$  intersect on the  $X$ -axis. In order to show this, we must return to the construction of the tangents at the points  $P_0$  and  $P_1$ . It was seen (Art. 45) that points  $B_0$  and  $B_1$  were found on the semi-circumferences  $P_0B_0Q_0$  and  $P_1B_1Q_1$  such that  $Q_0B_0=Q_1B_1$  ( $m=\mu$  in this case), and that then the lines  $P_0B_0$  and  $P_1B_1$  were the required tangents, which intersect on the  $X$ -axis.

CASE III.  $f_2(\mu) > 0$ .

A)  $f_2(y_0) \leq 0$ .

Then, as already shown,  $f_2(m)=0$  has two roots, one of which lies between 0 and  $\mu$ , and the other between  $\mu$  and  $y_0$ . Let these roots be  $m_1$  and  $m_2$  respectively. For the root  $m_1$ , we have



$$Q_0 T_0 = \frac{y_0 m_1}{\sqrt{y_0^2 - m_1^2}};$$

$$Q_1 T_1 = \frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}}.$$

We assert that here the intersection of the tangents at  $P_0$  and  $P_1$  lies on the other side of the  $X$ -axis from the curve.

In order to show this we need only prove that

$$Q_0 T_0 + Q_1 T_1 < Q_0 Q_1.$$

This is seen as follows:

$$f_2'(m_1) = -\frac{1}{m_1^2} \left[ \frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} + \frac{y_0 m_1}{\sqrt{y_0^2 - m_1^2}} - (x_1 - x_0) \right].$$

Now, since  $f_2'(m)$  within the interval  $0 \dots \mu$  is positive, and since  $m_1$  lies within this interval, it follows that  $f_2'(m_1)$  is positive. Therefore  $-(m_1)^2 f_2'(m_1)$  is negative, and consequently  $Q_0 T_0 + Q_1 T_1 - Q_0 Q_1$  is negative.

REMARK. In this consideration the whole interpretation depends upon the fact that the root lies in the interval  $0 \dots \mu$ , and the same discussion is applicable to Case B), where  $f_2(y_0) > 0$ , and where the root lies between  $0 \dots \mu$ .

52. On the consideration of the root  $m_2$ .

 1°. When  $f_2(y_0) \leq 0$ .

The root lies within the interval  $\mu \dots y_0$  and here  $f_2'(m)$  is negative within the interval; therefore  $-m^2 f_2'(m)$  is positive, and consequently

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0) > 0;$$

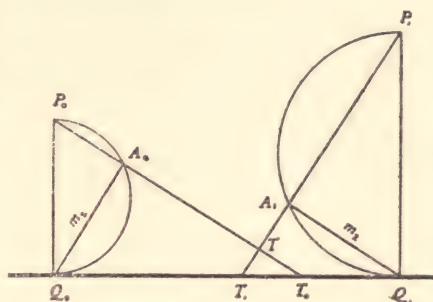
therefore

$$Q_0 T_0 + Q_1 T_1 > Q_0 Q_1;$$

so that  $T$  is on the same side of the  $X$ -axis as the curve.

2°. When  $f_2(y_0) > 0$ ; then the root  $m_2$  is a root of the equation  $f_1(m) = 0$ , so we have here to consider the sign of

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0)$$



within the interval  $0 \dots y_0$ .

We have proved that within this interval  $f_1'(m)$  is positive, and since

$$f_1'(m_2) = -\frac{1}{m_2^2} \left[ \frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0) \right]$$

is positive, it follows that

$$\left[ \frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0) \right]$$

is negative. Hence

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} < (x_1 - x_0).$$

Consequently

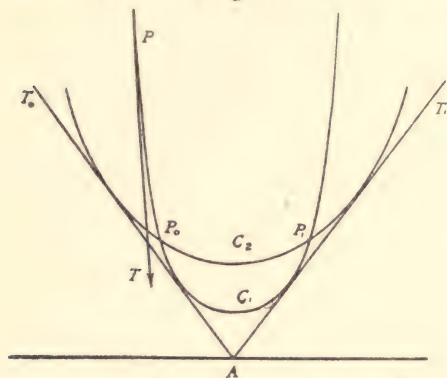
$$\frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - \frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} > (x_1 - x_0).$$

Since  $\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}}$  is a positive quantity, it follows *a fortiori* that

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} > (x_1 - x_0),$$

and the intersection lies on the same side of the  $X$ -axis as the curve.

53. We have seen that two catenaries having the same directrix cannot intersect in more than two points  $P_0$  and  $P_1$ . Denote as above the smaller parameter of these two curves by  $m_1$  and the larger by  $m_2$ . Then it is seen that  $C_1$ , the curve of smaller parameter, comes up from below and crosses  $C_2$ , the catenary of larger



parameter, and, having crossed  $C_2$ , never finds its way out again. For, consider the tangent  $PT$  to the curve  $C_1$  as the point  $P$  moves along this curve. This tangent must at first intersect  $C_2$ , but at the vertex it is parallel to the  $X$ -axis and evidently has no point in common with  $C_2$ . Hence, for some position between these two

positions the tangent to  $C_1$  must also be tangent to  $C_2$ . We see that there are two tangents common to  $C_1$  and  $C_2$ , and we shall next show that they intersect on the directrix.

54. Draw the common tangent  $AT_0$  and draw a tangent  $AT_1$  to the curve  $C_1$ . Then between these lines we may lay an infinite number of catenaries that have the same directrix. One of these catenaries must be  $C_2$ , for it touches  $AT_0$  and is the only catenary that can be drawn through the point of tangency made by  $AT_0$  (Art. 37). Consequently  $AT_1$  is the other common tangent to both curves.

We see also that the points  $P_0$  and  $P_1$  are beyond the points of contact of  $C_1$  with the two common tangents, while for  $C_2$  the points of contact of the tangents are beyond  $P_0$  and  $P_1$ . It is also seen that, as the two curves  $C_1$  and  $C_2$  tend to coincide, the common tangents to the distinct curve become tangents to the single curve at the points  $P_0$  and  $P_1$  (see Art. 51). If we call  $\mu$  the value of  $m$  corresponding to this latter curve, we have  $m_2 > \mu > m_1$ .

55. Suppose we have two catenaries which are not coincident and which have the same parameter  $m$ . Denote their equations by

$$y = m/2 [e^{(x-x_0')/m} + e^{-(x-x_0')/m}],$$

$$y = m/2 [e^{(x-x_0'')/m} + e^{-(x-x_0'')/m}].$$

These catenaries intersect in only one point. For we have at once

$$e^{(x-x_0')/m} + e^{-(x-x_0')/m} = e^{(x-x_0'')/m} + e^{-(x-x_0'')/m},$$

therefore

$$e^{x/m} [e^{-x_0'/m} - e^{-x_0''/m}] = e^{-x/m} [e^{x_0''/m} - e^{x_0'/m}],$$

or

$$e^{2x/m} = \frac{e^{x_0''/m} - e^{x_0'/m}}{e^{-x_0'/m} - e^{-x_0''/m}} = \frac{1}{e^{-(x_0' + x_0'')/m}} \left[ \frac{e^{x_0''/m} - e^{x_0'/m}}{-e^{x_0'/m} + e^{x_0''/m}} \right].$$

Therefore

$$e^{2x/m} = e^{(x_0' + x_0'')/m},$$

and consequently 
$$\begin{cases} x = (x_0' + x_0'')/2, \\ y = m/2 [e^{(x_0'' - x_0')/2m} + e^{-(x_0'' - x_0')/2m}], \end{cases}$$

which are the coordinates of *one* point.

### 56. Lindelöf's Theorem (1860).

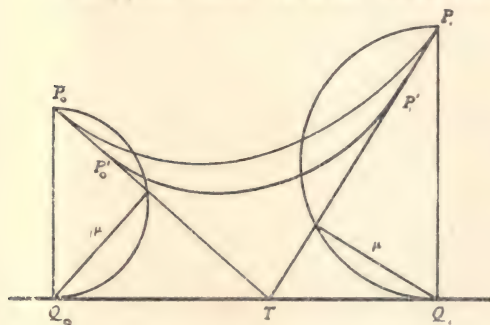
If we suppose the catenary to revolve around the  $X$ -axis, as also the lines  $P_0 T$  and  $P_1 T$ , then the surface-area generated by the revolution of the catenary is equal to the sum of the surface-areas generated by the revolution of the two lines  $P_0 T$  and  $P_1 T$  about the  $X$ -axis.

Suppose that with  $T$  as *center of similarity* (Ähnlichkeits-

punkt), the curve  $P_0 P_1$  is subjected to a strain so that  $P_0$  goes into the point  $P'_0$ , and  $P_1$  into the point  $P'_1$ , the distance  $P_0 P'_0$  being very small and equal, say, to  $\alpha = P_1 P'_1$ .

Then

$$P_0 T : P'_0 T = 1 : 1 - \alpha.$$



To abbreviate, let

$M_0$  denote the surface generated by  $P_0 T$ ;  $M'_0$  that generated by  $P'_0 T$ ;  $M_1$  denote the surface generated by  $P_1 T$ ;  $M'_1$  that generated by  $P'_1 T$ ;  $S$  that by the catenary  $P_0 P_1$ ;  $S'$  that by the catenary  $P'_0 P'_1$ .

From the nature of the strain, the tangents  $P_0 T$  and  $P_1 T$  are tangents to the new curve at the points  $P'_0$  and  $P'_1$ , so that we may consider  $P_0 P'_0 P'_1 P_1$  as a variation of the curve  $P_0 P_1$ .

It is seen that

$$S : S' = 1 : (1-a)^2;$$

$$M_0 : M'_0 = 1 : (1-a)^2;$$

$$M_1 : M'_1 = 1 : (1-a)^2.$$

Now from the figure we have as the surface of rotation of  $P_0 P'_0 P'_1 P_1$

$$(M_0 - M'_0) + S' + (M_1 - M'_1) + [(\alpha^2)] = S,$$

where  $[(\alpha^2)]$  denotes a variation of the second order.

Therefore

$$S - S' = (M_0 - M'_0) + (M_1 - M'_1) + [(\alpha^2)].$$

Hence

$$S[1 - (1-a)^2] = M_0[1 - (1-a)^2] + M_1[1 - (1-a)^2] + [(\alpha^2)],$$

and consequently

$$2aS = 2aM_0 + 2aM_1 + [(\alpha^2)],$$

or finally

$$S = M_0 + M_1,$$

a result which is correct to a differential of the first order.

In a similar manner

$$S' = M'_0 + M'_1;$$

so that

$$S - S' = (M_0 - M'_0) + (M_1 - M'_1);$$

or

$$S = (M_0 - M'_0) + S' + (M_1 - M'_1)$$

is an expression which is absolutely correct.

### 57. Another proof.

We have seen that

$$\frac{y_0 \mu}{\sqrt{y_0^2 - \mu^2}} + \frac{y_1 \mu}{\sqrt{y_1^2 - \mu^2}} - (x_1 - x_0) = 0, \quad [1]$$

and (see Fig. in Art. 45)

$$P_0 S = \frac{y_0^2}{\sqrt{y_0^2 - \mu^2}}; \quad P_1 S = \frac{y_1^2}{\sqrt{y_1^2 - \mu^2}}. \quad [2]$$

The surfaces of the two cones are, therefore, equal to

$$\frac{y_0 \cdot y_0^2 \pi}{\sqrt{y_0^2 - \mu^2}} \quad \text{and} \quad \frac{y_1 \cdot y_1^2 \pi}{\sqrt{y_1^2 - \mu^2}}.$$

The surface generated by the catenary is

$$\int_{x_0}^{x_1} 2 y \pi \, ds.$$

In the catenary  $ds = y/m \, dx$  (see Art. 35), so that

$$\begin{aligned} \int_{x_0}^{x_1} 2 y \pi \, ds &= \int_{x_0}^{x_1} (2 y^2 \pi \, dx) / m \\ &= 2\pi \int_{x_0}^{x_1} m^2 / 4 [e^{2(x-x_0')/m} + 2 + e^{-2(x-x_0')/m}] dx / m \\ &= \frac{1}{4} \pi m^2 [e^{2(x-x_0')/m} - e^{-2(x-x_0')/m} + 4x/m]_{x_0}^{x_1} \\ &= \pi [\frac{1}{2} m (e^{(x-x_0')/m} + e^{-(x-x_0')/m}) \cdot \frac{1}{2} m (e^{(x-x_0')/m} - e^{-(x-x_0')/m}) + mx]_{x_0}^{x_1} [A] \\ &= \pi [\pm y \sqrt{y^2 - m^2} + mx]_{x_0}^{x_1} \\ &= \pi [y_1 \sqrt{y_1^2 - m^2} + y_0 \sqrt{y_0^2 - m^2} + m(x_1 - x_0)], \quad [B] \end{aligned}$$

where we have taken the + sign with  $y_0 \sqrt{y_0^2 - m^2}$  because  $x_0 - x_0'$  is negative, hence  $e^{(x-x_0')/m} - e^{-(x-x_0')/m}$  in [A] is negative.

But from [1]

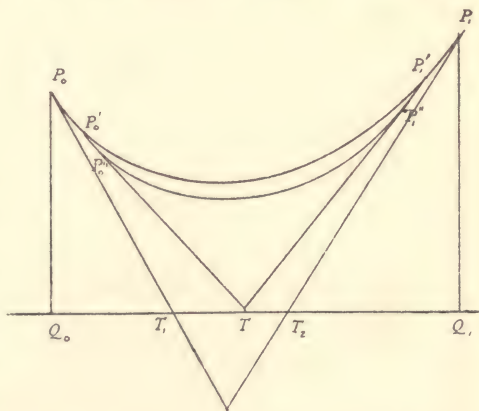
$$x_1 - x_0 = \frac{y_1 \mu}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0 \mu}{\sqrt{y_0^2 - \mu^2}}.$$

Substituting in  $[B]$ , we have, after making  $m = \mu$ , for the area generated by the revolution of the catenary

$$\begin{aligned} \pi \left[ y_1 \sqrt{y_1^2 - \mu^2} + \frac{y_1 \mu^2}{\sqrt{y_1^2 - \mu^2}} + y_0 \sqrt{y_0^2 - \mu^2} + \frac{y_0 \mu^2}{\sqrt{y_0^2 - \mu^2}} \right] \\ = \pi \left[ \frac{y_1^3}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0^3}{\sqrt{y_0^2 - \mu^2}} \right], \end{aligned}$$

which, as shown above, is the sum of the surface-areas of the two cones.

58. Let us consider\* again the following figure, in which the strain is represented. In order to have a minimum surface of revolution, the curve which we rotate must satisfy the differential equation of the problem. If, then, we had a minimum, this would be brought about by the rotation of the catenary; for the catenary is the curve which satisfies the differential equation. But in our figure this curve can produce no minimal surface of revolution for two reasons: 1<sup>o</sup> because, drawing tangents (in Art. 59 it is proved that there exists an infinite number) which intersect on the  $X$ -axis, it is seen that the rotation of  $P'_0 P'_1$  is the same as that of the two lines  $P'_0 T$  and  $P'_1 T$ , as shown above, so that there are an infinite number of lines that may be drawn between  $P_0$  and  $P_1$  which give the same surface of revolution as the catenary between these points; 2<sup>o</sup> because between  $P_0$  and  $P_1$  lines may be drawn which, when caused to revolve about the  $X$ -axis, would produce a smaller surface-area than that produced by the revolution of the catenary. For the surface-area generated by the revolution of  $P'_0 P'_1$  is the same as that generated by  $P'_0 P''_0 P''_1 P'_1$ . But the straight lines  $P'_0 P''_0$  and  $P'_1 P''_1$  do not satisfy the differential equation of the problem, since they are not catenaries. Hence the first variation along these lines is  $\geq 0$ , so that between the points



\* See also Todhunter, *Researches in the Calculus of Variations*, p. 29.

$P_0', P_0''$  and  $P_1', P_1''$  curves may be drawn whose surface of rotation is smaller than that generated by the straight lines  $P_0'P_0''$  and  $P_1'P_1''$ .

The Case II, given above and known as the transition case, *i. e.*, where the point of intersection of the tangents pass from one side to the other side of the  $X$ -axis, affords also no minimal surface, since, as already seen, there are, by varying the quantity  $a$  (Art. 56), an infinite number of surfaces of revolution that have the same area.

59. In Case III we had two roots of  $m$ , which we called  $m_1$  and  $m_2$ , where  $m_2 > m_1$ . We consider first the catenary with parameter  $m_1$ . This parameter satisfies the inequality

$$\frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} + \frac{y_0 m_1}{\sqrt{y_0^2 - m_1^2}} < x_1 - x_0. \quad [A]$$

The equation of the tangent to the curve is

$$\frac{dy}{dx} = \frac{y' - y}{x' - x},$$

where  $x'$  and  $y'$  are the running coordinates. The intersection of this line with the  $X$ -axis is

$$x' - x = -\frac{y}{dy/dx}, \quad \text{or} \quad x' = x - \frac{y}{dy/dx};$$

*i. e.*,

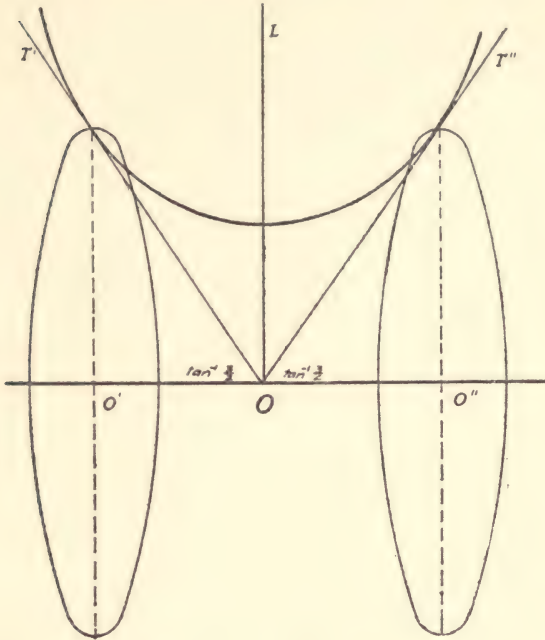
$$x' = x - m \left[ \frac{e^{(x-x_0')/m} + e^{-(x-x_0')/m}}{e^{(x-x_0')/m} - e^{-(x-x_0')/m}} \right].$$

Hence, when  $x = x_0'$ ,  $x' = -\infty$ , and when  $x = +\infty$ ,  $x' = +\infty$ .

On the other hand,  $dx'/dx$  is always positive, so that  $x'$  always increases when  $x$  increases, and the tangent passes from  $-\infty$  along the  $X$ -axis to  $+\infty$ , and never passes twice through the same point. It is thus seen that there are an infinite number of pairs of points on the catenary between the points  $P_0$  and  $P_1$  such that the tangents at any of these pairs of points intersect on the  $X$ -axis, and there can consequently be no minimum. Such pairs of points are known as *conjugate points*.

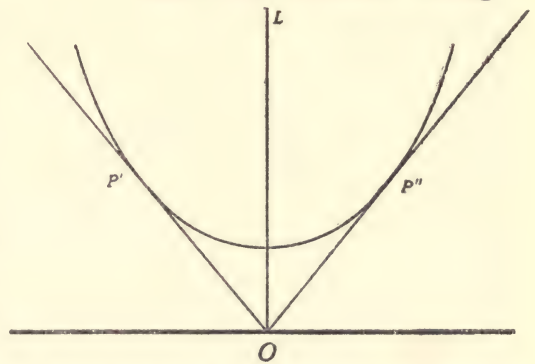
When  $m = m_2$ , the tangents intersect above the  $X$ -axis, and there is in reality a minimum, as will be seen later.

60. *Application.* Suppose we have two rings of equal size attached to the same axis which passes perpendicularly through their centers. If the rims of these rings are connected by a free film of liquid (soap solution), what form does the film take?



By a law in physics the film has a tendency to make its area as small as possible. Hence, only as a minimal surface will the film be in a state of equilibrium. Let  $O$  be midway between  $O'$  and  $O''$ . The film is symmetric with respect to the  $OO''$  and  $OL$  axes and has the form of a surface of revolution about the  $OO''$  axis, this surface being a

catenoid. The line  $OL$  is the axis of symmetry of the generating catenary. Construct the tangents  $OP''$  and  $OP'$  from the origin to the catenary. Only when  $P'$  and  $P''$  are situated beyond the rims of the circles will the generating arc of the catenary be free from conjugate points, and only then will we have a minimal surface and a position of stable equilibrium of the film.



61. We saw (Art. 38) that all catenaries having the same axis of symmetry and the same directrix may be laid between two lines inclined approximately at an angle  $\tan^{-1} (3/2)$  to the directrix and which pass through the intersection of the directrix and the axis of symmetry. All catenaries under consideration then are ensconced within the lines  $OP'$  and  $OP''$  and have these lines as tangents. The arcs of these catenaries between their points of con-

tact with  $OT'$  and  $OT''$  do not intersect one another. Through any point  $P_0$  inside the angle  $T'OT''$  will evidently pass one of these arcs, and the same arc (on account of the axis of symmetry  $OL$  of the catenary) will contain the point  $P_1$  symmetrical to  $P_0$  on the other side of  $OL$ . The arc  $P_0P_1$  contains no conjugate point (Chap. IX, Art. 128), and therefore generates a minimal surface of revolution. Further, this is the only arc of a catenary through the points  $P_0$  and  $P_1$  which generates a minimal surface.

Suppose that we started out with our two rings in contact and shoved them along the axis at the same rate and in opposite directions from the point  $O$ . As long as  $P_0$  and  $P_1$  are situated within the angle  $T'OT''$  (or what is the same thing, as long as  $P_0OP_1 < T'OT''$ ) then the tangents at  $P_0$  and  $P_1$  meet on the upper side of the  $X$ -axis and there exists an arc of a catenary which gives a minimal surface of revolution and the film has a tendency to take a definite position and hold itself there. But as soon as the angle  $P_0OP_1$  becomes equal to or greater than  $T'OT''$  this tendency ceases and the equilibrium of the film becomes unstable. As a matter of fact (see Art. 101), the only minimum which now exists is that given by the surface of the two rings, the film having broken and gone into this form.

## CHAPTER IV.

PROPERTIES OF THE FUNCTION  $F(x, y, x', y')$ .

62. Consider the general integral of Art. 13:

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt,$$

where  $F$  is a given function of the four arguments,  $x, y, x', y'$ , the quantities  $x'$  and  $y'$  being written for  $dx/dt$  and  $dy/dt$ ; further we must regard  $F$  as a *one-valued* regular function of these four arguments, one-valued not in the analytical sense, but only for real values of the arguments;  $x$  and  $y$  are defined for the whole plane or for a connected portion of it, while  $x'$  and  $y'$  are to be considered as variables that are not limited, since they determine the direction of the tangent, and it is supposed that we may go in any direction from the point  $x, y$ . In our problem new assumptions are made regarding  $x$  and  $y$ , but not regarding  $x'$  and  $y'$ .<sup>\*</sup> We further assume that the functions  $x, y, x'$  and  $y'$ , are capable of being differentiated, and that the curve is regular throughout its whole extent, or is composed of regular portions. Consequently  $x$  and  $y$  considered as functions of  $t$  and written  $x(t), y(t)$  are one-valued regular functions of  $t$  throughout its whole extent or throughout the regular portions; in the latter case we shall limit ourselves to one regular portion. If we did not make this assumption, the curve could not be the subject of mathematical investigation, since there is no method of treating irregular curves in their generality; and, if we wish the rules of the differential and integral

<sup>\*</sup> A limitation has to be made, however, if for certain values of  $x', y'$  the function  $F$  becomes infinitely large. Such cases must be excluded from the present discussion.

calculus to be sufficient, then we must first apply our investigation to such functions, to which the rules are applicable without any limitation; that is, to functions having the above properties.

63. If we find a curve which is regular and which satisfies the conditions of the problem, then it still remains as a supplement to prove that it is the only curve which satisfies the conditions of the problem.

For example, it is found that of all regular closed curves of given perimeter the circle is the one which encloses the greatest surface-area; *à priori*, however, it is not known that a regular curve satisfies the problem. We know that of all polygons with a given number of sides and having a given perimeter the regular polygon has the greatest surface-area, and we thus come to the conclusion that the circle, to which the polygon approaches when the number of sides is increased, will have the greatest surface-area of all the closed curves; however, no one will recognize in this a rigorous proof, and in fact there still remains a peculiar artifice to prove this property of the circle.

64. The chief difficulty in all analysis consists in giving a strenuous proof that the necessary conditions, that have been found for the existence of a certain property, are also sufficient. In analytical researches we make conclusions in the following manner: If the analytical quantities exist, which are required through the problems that have been set, then they must have certain properties; this gives the necessary conditions for the sought functions. It remains yet reciprocally to prove: If the conditions for an analytical object (curve, surface, etc.) are fulfilled, then the analytical object satisfies the conditions of the problem.

We therefore presuppose in our investigations, that the required functions are regular in their whole extent, and we seek the necessary conditions for the function which are given from the problems. Finally we will free ourselves from the limitations as far as it is possible, and see whether also the functions which have been found correspond to the conditions of the problem.

65. The development of a mathematical idea is, as a rule, first suggested by a concrete instance. We assume, for example, the existence in nature of something which we call the *area* of a limited plane. This area we express by a mathematical formula.

We extend our formula and talk of the area of a curved surface. The mathematical formula exists. That to which it corresponds in nature may or may not have an objective existence. The word "area," however, is defined for us, and is limited by the mathematical formula. When the formula ceases to be intelligible, ceases to have a meaning and to give a value, then also does the idea "area" cease to exist for us. We must always presuppose those limitations to be involved in our symbols which permit of the formula having a meaning.

66. Only for regular curves do we compare our integrals; for such curves alone have they a meaning. Among this class of curves we seek one which gives a maximum or a minimum value of our integral. And when we put our theory into practice we assume the *non-existence* of quantities other than those which our theory has actually compared. Here we run a risk.

It may be that in some particular problem we have assigned a certain role; it may be also that, as far as our theory goes, we are correct in assuming the possibility of the existence of all the regular curves that are compared with one another and that their roles relative to one another has not been misstated. But it may be that there exists in nature the possibility of quantities other than those defined by our definite integrals along regular curves, and these quantities may have the same essential properties relative to the problem in question as our various definite integrals. It may be also that to one of these quantities nature has assigned that very role which we have been seeking among our definite integrals.

When we apply any mathematical theory to objective reality, we make assumptions in the way of continuity, differentiation, etc., regarding the possibilities which are permitted in nature. The question arises, do our hypotheses include all possibilities?

67. We may emphasize the fact that in the development of a general theory, as a rule its scope is not determined beforehand. The quantities and functions to which we must apply the operations involved are named *à priori*, but formulæ are developed on the supposition that the operations involved are feasible and have a meaning. The scope of the formulæ is afterwards defined by the territory in which all the steps involved have some significa-

tion, or by the exclusion of any realm in which they would be incapable of interpretation.

68. We will now prove some important properties of the function  $F$  (Art. 62). In the problems which we have discussed the following is to be observed: the value of the integral, which is to be a minimum, depends in all cases only upon the form of the curve which is to be determined, not upon the manner in which  $x, y$  are represented as functions of a quantity  $t$ .

For example, if in the first problem we write the integral in the form

$$\int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

then  $t$  is exactly equal to  $x$ , and it is clear that the value of this integral is the same as it was for the previous form (Art. 7).

If we write for  $t$  any function of another quantity  $\tau$  of such a nature that to the values  $t_0$  and  $t_1$  of  $t$  the values  $\tau_0$  and  $\tau_1$  of  $\tau$  correspond, and that the curve with increasing  $\tau$  will be traversed in the same direction as in the first case with increasing  $t$ , then the integral must remain unaltered, if it is to be independent of the manner in which  $x, y$  are represented as functions of the quantity  $t$ ; that is, we must have as the integral  $I$  of Art. 62

$$1) \quad \int_{t_0}^{t_1} F\left\{x, y, \frac{dx}{dt}, \frac{dy}{dt}\right\} dt = \int_{\tau_0}^{\tau_1} F\left\{x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right\} d\tau.$$

The simplest function of this kind that we can write for  $t$ , is  $t = k\tau$ , where  $k$  represents any arbitrary but *positive* quantity. Hence considering  $x, y$  as functions of  $\tau$  in the left-hand side of 1), we have

$$\int_{t_0}^{t_1} F\left\{x, y, \frac{dx}{dt}, \frac{dy}{dt}\right\} dt = \int_{\tau_0}^{\tau_1} F\left\{x, y, \frac{1}{k} \frac{dx}{d\tau}, \frac{1}{k} \frac{dy}{d\tau}\right\} k d\tau;$$

hence

$$2) \quad \int_{\tau_0}^{\tau_1} F \left\{ x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau} \right\} d\tau = \int_{\tau_0}^{\tau_1} F \left\{ x, y, \frac{1}{k} \frac{dx}{d\tau}, \frac{1}{k} \frac{dy}{d\tau} \right\} k d\tau.$$

Since this equation must be true for any arbitrary positive value of  $k$ , which however is not necessarily a constant, but may be any continuous positive function, it follows that the functions to be integrated must themselves be equal for every positive value of  $k$ ; and consequently

$$F \left( x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau} \right) = k F \left( x, y, \frac{1}{k} \frac{dx}{d\tau}, \frac{1}{k} \frac{dy}{d\tau} \right);$$

or, writing  $k=1/k$ :

$$3) \quad F(x, y, kx', ky') = k F(x, y, x', y').$$

That is, if the integral  $I$  is to depend only upon the form of the curve (or in other words, upon the analytical connection between  $x$  and  $y$ ), then  $F(x, y, x', y')$ , with regard to  $x'$  and  $y'$  must be a homogeneous function of the first degree. This condition is also sufficient to assure that the integral depends only upon the form of the curve; for consider  $x, y$  first expressed as functions of a quantity  $t$  and then as functions of a quantity  $\tau$ , and if these functions are of such a nature that the curve is traversed from the beginning-point to the end-point when  $t$  takes all values from  $t_0$  to  $t_1$ , and  $\tau$  all values  $\tau_0$  to  $\tau_1$ , then we can write  $dt/d\tau = k$ , if  $t$  increases at the same time as  $\tau$ . Since  $k$  is a positive quantity, the correctness of the expression 2) follows from the existence of 3) and at the same time also the correctness of 1).

It follows also that

$$\begin{aligned} \frac{\partial F(x, y, kx', ky')}{\partial(kx')} &= k \frac{\partial F(x, y, x', y')}{\partial(kx')} = \frac{\partial F(x, y, x', y')}{\partial x'} \\ &= F^{(1)}(x, y, x', y'), \text{ say.} \end{aligned}$$

In the same way the partial derivative of  $F$  with respect to its fourth argument is invariative and may be denoted by  $F^{(2)}(x, y, x', y')$ .

69. The condition that  $F(x, y, x', y')$  must be a homogeneous function of the first degree with regard to  $x'$  and  $y'$  is generally expressed in another manner. In fact, it is nothing else than the condition of integrability of  $F$ . For if  $F(x, y, x', y') dt$  is to be an exact differential, so that, say,  $F(x, y, x', y') = d\phi$ , then the equation

$F(x, y, x', y') = (\partial\phi/\partial x)x' + (\partial\phi/\partial y)y' + (\partial\phi/\partial x')x'' + (\partial\phi/\partial y')y''$  must exist identically.

Since no second differential quotient is present in  $F$ , it follows that  $\partial\phi/\partial x' = 0$  and  $\partial\phi/\partial y' = 0$ , i. e.,  $\phi$  does not contain explicitly  $x'$  or  $y'$ , and therefore

$$F(x, y, x', y') = (\partial\phi/\partial x)x' + (\partial\phi/\partial y)y'.$$

But this is nothing more than that  $F$  is a homogeneous function of the first degree in  $x', y'$ .

This is everywhere the case in the examples given in Chap. I.

70. If the curve is of such a nature that one may regard the one coordinate as a one-valued function of the other and in such a way that for every value of  $x$  between two limits  $x_0$  and  $x_1$ , there corresponds only one definite value of  $y$ , and that  $x$  continuously increases when we traverse the curve from the beginning-point to the end-point, then we may choose for  $t$  the quantity  $x$  itself, and therefore write the integral in the form

$$4) \quad I = \int_{x_0}^{x_1} F(x, y, 1, dy/dx) dx,$$

as it is usually written.

71. This representation is not always true, since the above conditions which are necessary are not always fulfilled; for example, in the fourth problem of Chapter I we must distribute the yet unknown curve into several parts, and this is not always convenient.

On the other hand, a representation such as given above is always possible, if we introduce the quantity  $l$ , since one could introduce as the variable  $l$  the arc  $s$  of the curve measured from the

beginning-point. Besides in the form 4) it sometimes unavoidably happens that  $dy/dx$  and consequently  $F$  becomes infinite within the limits of integration; on the other hand it is generally possible so to choose  $t$  that this is not the case.

For these reasons, in spite of the fact that many developments become more cumbrous, it is preferable to treat the integral  $I$  in the form

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt;$$

for on the other hand its great symmetry overbalances the fault just mentioned.

72. *Analytical condition for  $F(x, y, x', y')$ .*

In the relation (Art. 68)

$$F(x, y, kx', ky') = k F(x, y, x', y'),$$

write  $k=1+h$ , then is

$$F[x, y, (1+h)x', (1+h)y'] = (1+h) F(x, y, x', y'),$$

or,

$$F + \left( \frac{\partial F}{\partial x'} x' + \frac{\partial F}{\partial y'} y' \right) h + h^2(\dots) = F + h F,$$

where

$$F \equiv F(x, y, x', y').$$

Therefore equating the coefficients of  $h$ :

$$F = x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'}, \quad [1]$$

which again is the condition of homogeneity.

73. Differentiate the above equation [1] first with regard to  $x'$  and then with regard to  $y'$ , which is allowable, since  $F$  is a regular function, and  $x', y'$  vary in a continuous manner, and we have

$$\begin{aligned} \alpha) & \quad \frac{\partial^2 F}{\partial x'^2} x' + \frac{\partial^2 F}{\partial y' \partial x'} y' = 0, \\ \beta) & \quad \frac{\partial^2 F}{\partial x' \partial y'} x' + \frac{\partial^2 F}{\partial y'^2} y' = 0. \end{aligned} \quad \left\{ \dots \right. \quad [2]$$

Hence, from  $\alpha$ )

$$\frac{\partial^2 F}{\partial x'^2} : \frac{\partial^2 F}{\partial y' \partial x'} = y' : -x' = y'^2 : -x' y',$$

and from  $\beta$ )

$$\frac{\partial^2 F}{\partial x' \partial y'} : \frac{\partial^2 F}{\partial y'^2} = -y' : x' = -x' y' : x'^2;$$

therefore

$$\frac{\partial^2 F}{\partial x'^2} : \frac{\partial^2 F}{\partial x' \partial y'} : \frac{\partial^2 F}{\partial y'^2} = y'^2 : -x' y' : x'^2;$$

and, if  $F_1$  denotes the factor of proportionality, we have:

$$\frac{\partial^2 F}{\partial x'^2} = F_1 y'^2; \quad \frac{\partial^2 F}{\partial x' \partial y'} = -F_1 x' y'; \quad \frac{\partial^2 F}{\partial y'^2} = F_1 x'^2, \quad [3]$$

and consequently

$$\frac{\frac{\partial^2 F}{\partial x'^2}}{y'^2} = \frac{\frac{\partial^2 F}{\partial x' \partial y'}}{-x' y'} = \frac{\frac{\partial^2 F}{\partial y'^2}}{x'^2} = F_1.$$

$F$  is of the first dimension in  $x'$  and  $y'$ ;  $\frac{\partial F}{\partial x'}$ ,  $\frac{\partial F}{\partial y'}$  are of the dimension 0 in  $x'$  and  $y'$ ;  $\frac{\partial^2 F}{\partial y'^2}$ ,  $\frac{\partial^2 F}{\partial x' \partial y'}$ ,  $\frac{\partial^2 F}{\partial x'^2}$  are of the  $-1$ st dimension in  $x'$  and  $y'$ ; consequently  $F_1$  is of the dimension  $-3$  in  $x'$  and  $y'$ .

This function  $F_1$  plays an exceedingly important rôle in the whole theory.

## CHAPTER V.

## THE VARIATION OF CURVES EXPRESSED ANALYTICALLY.

## THE FIRST VARIATION.

74. In Chapter II we considered examples of special variations. The method followed provided for the displacement of a curve in one direction only, in the direction parallel to the  $X$ -axis, and is consequently applicable only to the comparison of integrals along curves obtained from one another by such a deformation.

We shall now give a more general form to the variations employed and shall seek strenuous methods for the solution of the general problem of variations proposed in Chapter I. After deriving the necessary conditions we shall then proceed to discuss the *sufficient* conditions. In order to develop the conditions for the appearance of a maximum or a minimum of the integral

$$1) \quad I = \int_{t_0}^{t_1} F(x, y, x', y') dt,$$

it is necessary to study more closely the conception of the variation of a curve and fix this conception analytically.

By writing instead of each point  $x, y$  of a curve (presupposed regular) another point  $x + \xi, y + \eta$ , we transform the first curve into another regular curve. This second curve is neighboring the first curve if we make sufficiently small the quantities  $\xi$  and  $\eta$  which like  $x$  and  $y$  we consider as one-valued continuous functions of  $t$ .

75. The following is one of the methods of effecting this result. Let  $\xi$  and  $\eta$  be continuous functions of  $t$  and also of a quan-

tity  $k$ . We further suppose that  $\xi$  and  $\eta$  vanish when  $k=0$  for every value of  $t$ , for example

$$\xi = k u(t); \quad \eta = k v(t),$$

$u$  and  $v$  being finite and continuous functions of  $t$ .

The functions  $u$  and  $v$  and consequently also  $\xi$  and  $\eta$  are subject to further conditions. It is in general required to construct a curve between two given points which first may be regarded as fixed. Later the condition of their variability may be introduced. Consequently we have to consider only such values of  $t$  that  $\xi$ ,  $\eta$  and consequently  $u$ ,  $v$  vanish on the limits.

If for  $x, y$  we write  $x+\xi, y+\eta$ , then for  $x', y'$  we must write  $x'+\xi', y'+\eta'$ . Further, the function  $F(x+\xi, y+\eta, x'+\xi', y'+\eta')$  must be developed in powers of  $\xi, \eta, \xi'$  and  $\eta'$ . For the convergence of this series, it is necessary that  $\xi, \eta, \xi'$  and  $\eta'$  have finite values.

Now, if we write

$$\xi = k \sin t/k^n; \quad \eta = k \cos t/k^n,$$

then we have

$$\frac{d\xi}{dt} = k^{1-n} \cos t/k^n; \quad \frac{d\eta}{dt} = -k^{1-n} \sin t/k^n,$$

so that, whereas  $\xi$  and  $\eta$  have infinitely small values for infinitely small values of  $k$ , the quantities  $\frac{d\xi}{dt}, \frac{d\eta}{dt}$  vacillate for  $n=1$  between  $+1$  and  $-1$  and become infinite for  $n>1$ . We shall consequently consider only such special variations in which  $u$  and  $v$  are functions of  $t$  alone, and which with their derivatives are finite and continuous between the limits  $t_0$  and  $t_1$ . We thus restrict, in a great measure, the arbitrariness of the indefinitely small variations of the curve, and thus exclude a great many *neighboring* curves from the discussion. However, there exist among all the possible neighboring curves also such which satisfy the above conditions, and with these we shall first establish the necessary conditions and later show that the necessary conditions thus established are also sufficient for the establishment of the existence of a maximum or a minimum value of the integral. (See Arts. 134 et seq.)

76. We have instead of *one* neighboring curve a whole bundle of such curves if we make the substitutions

$$\begin{array}{c|c} x & x + \epsilon \xi, \\ y & y + \epsilon \eta, \\ x' & x' + \epsilon \xi', \\ y' & y' + \epsilon \eta', \end{array}$$

and let  $\epsilon$ , a quantity independent of the variables in  $F(x, y, x', y')$ , vary between  $+1$  and  $-1$ .

The total variation that is thereby introduced in the integral of the preceding article is

$$I + \Delta I = \int_{t_0}^{t_1} F(x + \epsilon \xi, y + \epsilon \eta, x' + \epsilon \xi', y' + \epsilon \eta') dt,$$

which developed by Maclaurin's Theorem is

$$= \int_{t_0}^{t_1} \left\{ F + \epsilon \left[ \frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial x'} \xi' + \frac{\partial F}{\partial y'} \eta' \right] + \epsilon^2 (\dots) \right\} dt.$$

Further (see Art. 25)

$$\Delta I = \epsilon \delta I + \frac{\epsilon^2}{1.2} \delta^2 I + \dots;$$

hence, equating coefficients of  $\epsilon$ ,

$$2) \quad \delta I = \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial x'} \xi' + \frac{\partial F}{\partial y'} \eta' \right] dt.$$

But

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'} \cdot \frac{d\xi}{dt} \cdot dt = \left[ \frac{\partial F}{\partial x'} \cdot \xi \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \xi \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) dt;$$

so that 2) becomes

$$2') \quad \delta I = \int_{t_0}^{t_1} \left\{ \left[ \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) \right] \xi + \left[ \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right) \right] \eta \right\} dt \\ + \left[ \frac{\partial F}{\partial x'} \cdot \xi + \frac{\partial F}{\partial y'} \cdot \eta \right]_{t_0}^{t_1};$$

or

$$3') \quad \delta I = \int_{t_0}^{t_1} \left\{ G_1 \xi + G_2 \eta \right\} dt + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t_0}^{t_1},$$

where

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right),$$

$$G_2 = \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right).$$

77. Owing to the hypotheses that  $x', y', \frac{d\xi}{dt}, \frac{d\eta}{dt}$  vary in a continuous manner with  $t$  [that is, within the portion of curve considered no sudden change enters in the direction of the curve], the first variation of the integral  $I$  may be transformed in a remarkable manner.

We had

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right)$$

and also (Art. 72)

$$F(x, y, x', y') = x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'}.$$

Therefore

$$\frac{\partial F}{\partial x} = x' \frac{\partial^2 F}{\partial x' \partial x} + y' \frac{\partial^2 F}{\partial y' \partial x},$$

and differentiating  $\frac{\partial F}{\partial x'}$  with respect to  $t$ , we have

$$-\frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) = -\frac{\partial^2 F}{\partial x \partial x'} \frac{dx}{dt} - \frac{\partial^2 F}{\partial y \partial x'} \frac{dy}{dt} - \frac{\partial^2 F}{\partial x'^2} \frac{dx'}{dt} - \frac{\partial^2 F}{\partial y' \partial x'} \frac{dy'}{dt}.$$

Hence,

$$G_1 = y' \left( \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'} \right) - \left( \frac{\partial^2 F}{\partial x'^2} \frac{dx'}{dt} + \frac{\partial^2 F}{\partial y' \partial x'} \frac{dy'}{dt} \right).$$

Writing  $\frac{\partial^2 F}{\partial x'^2} = y'^2 F_1$  (Art. 73), and  $\frac{\partial^2 F}{\partial y' \partial x'} = -x' y' F_1$ , and defining  $G$  by the equation

$$G = \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'} - F_1 \left( y' \frac{dx'}{dt} - x' \frac{dy'}{dt} \right),$$

it is seen that

$$G_1 = y' G.$$

In a similar manner it may be shown that

$$G_2 = -x' G.$$

78. *Lemma.* If  $\phi(t)$  and  $\psi(t)$  are two continuous functions of  $t$  between the limits  $t_0$  and  $t_1$  and if the integral

$$\int_{t_0}^{t_1} \phi(t) \psi(t) dt$$

is always zero, in whatever manner  $\psi(t)$  is chosen, then necessarily  $\phi(t)$  must vanish for every value of  $t$  between  $t_0$  and  $t_1$ .

The following proof, due to Prof. Schwarz, is a geometrical interpretation of a method due to Heine.\*

Suppose it possible that the function  $\phi(t)$  has a finite value for a point  $t=t'$  situated between  $t_0$  and  $t_1$ . Then owing to the continuity of  $\phi(t)$  we can find an interval  $t'-d \dots t'+d$  within which  $\phi(t)$  also has a finite value.

We write the integral in the form

$$\begin{aligned} \int_{t_0}^{t_1} \phi(t) \psi(t) dt &= \int_{t_0}^{t'-d} \phi(t) \psi(t) dt + \int_{t'-d}^{t'+d} \phi(t) \psi(t) dt \\ &\quad + \int_{t'+d}^{t_1} \phi(t) \psi(t) dt. \end{aligned}$$

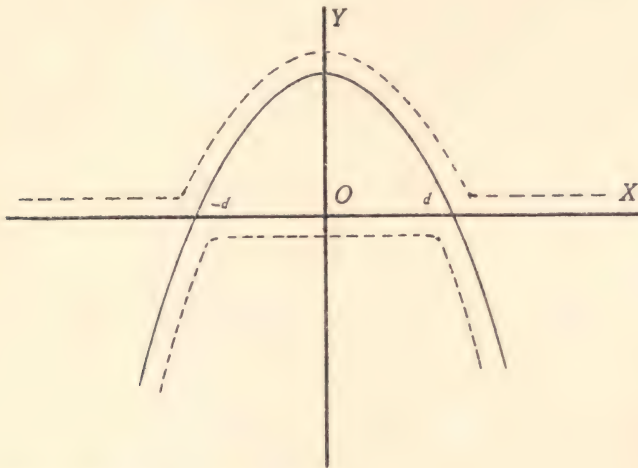
\* Heine, Crelle, bd. 54, p. 338.

The second integral on the right hand side may be written

$$M \int_{t'-d}^{t'+d} \psi(t) dt,$$

where  $M$  is the mean value of  $\phi(t)$  for a value of  $t$  within the interval  $t'-d \dots t'+d$ .

We shall show that it is possible to determine a function  $\psi(t)$  which will render this integral positive and greater than the sum of the first and third integrals in the above expression, while at the same time  $\psi(t_0) = \psi(t_1) = 0$ .



Let us form the equation

$$\left\{ y-1 + \frac{x^2}{d^2} \right\} y=0,$$

which represents the parabola  $y=1 - \frac{x^2}{d^2}$  and the  $X$ -axis.

Consider next the equation

$$\left\{ y-1 + \frac{x^2}{d^2} \right\} y=\epsilon^2,$$

where  $\epsilon$  is a small quantity. By taking  $\epsilon$  sufficiently small, this curve can be made to approach as near as we wish the parabola and the  $X$ -axis.

Solving the above equation for  $y$ , we have as the two roots (two branches)

$$y = \frac{1}{2} \left\{ 1 - \frac{x^2}{d^2} \right\} \pm \sqrt{\frac{1}{4} \left\{ 1 - \frac{x^2}{d^2} \right\}^2 + \epsilon^2}.$$

The branch

$$y = \frac{1}{2} \left( 1 - \frac{x^2}{d^2} \right) + \sqrt{\frac{1}{4} \left( 1 - \frac{x^2}{d^2} \right)^2 + \epsilon^2}$$

is symmetrical with respect to the  $Y$ -axis, and for values of  $x$ , such that  $-d \leq x \leq +d$ , the ordinate of any point of the curve is greater than the corresponding ordinate of the parabola.

For the parabola  $y = 1 + \frac{x^2}{d^2}$ , the integral

$$\int_{-d}^{+d} y dx = 4/3 d.$$

It follows then that for the curve we must have

$$\int_{-d}^{+d} \left[ \frac{1}{2} \left( 1 - \frac{x^2}{d^2} \right) + \sqrt{\frac{1}{4} \left( 1 - \frac{x^2}{d^2} \right)^2 + \epsilon^2} \right] dx > \frac{4}{3} d;$$

for  $|x| = d$ , we have  $y = \epsilon$ ; and from the inequality

$$\begin{aligned} y &= -\frac{1}{2} \left( \frac{x^2}{d^2} - 1 \right) + \sqrt{\frac{1}{4} \left( \frac{x^2}{d^2} - 1 \right)^2 + \epsilon^2} \\ &< -\frac{1}{2} \left( \frac{x^2}{d^2} - 1 \right) + \frac{1}{2} \left( \frac{x^2}{d^2} - 1 \right) + \epsilon, \end{aligned}$$

it follows for  $|x| > d$ , that  $y$  is positive and  $< \epsilon$ . For the lower branch  $y$  is negative and the curve

$$y = -\frac{1}{2} \left\{ \frac{x^2}{d^2} - 1 \right\} - \sqrt{\frac{1}{4} \left\{ \frac{x^2}{d^2} - 1 \right\}^2 + \epsilon^2}$$

follows the parabola to infinity as shown in the figure. It is however the upper branch which we use, since  $y$  is less than  $\epsilon$ , as soon as  $x$  passes the value  $d$  on either side of the origin.

Instead of the integral last written, take the integral which has the same value

$$\int_{t'-d}^{t'+d} \left( \frac{1}{2} \left\{ 1 - \frac{(t-t')^2}{d^2} \right\} + \sqrt{\frac{1}{4} \left\{ 1 - \frac{(t-t')^2}{d^2} \right\}^2 + \epsilon^2} \right) dt > \frac{4}{3} d.$$

Writing

$$\chi(t) = \frac{1}{2} \left\{ 1 - \frac{(t-t')^2}{d^2} \right\} + \sqrt{\frac{1}{4} \left\{ 1 - \frac{(t-t')^2}{d^2} \right\}^2 + \epsilon^2},$$

we have

$$\begin{aligned} \int_{t_0}^{t_1} \phi(t) \chi(t) dt &= \int_{t_0}^{t'-d} \phi(t) \chi(t) dt + \int_{t'-d}^{t'+d} \phi(t) \chi(t) dt + \int_{t'+d}^{t_1} \phi(t) \chi(t) dt \\ &= M' \int_{t_0}^{t'-d} \chi(t) dt + M \int_{t'-d}^{t'+d} \chi(t) dt + M'' \int_{t'+d}^{t_1} \chi(t) dt \\ &= M' [< \epsilon] (t' - d - t_0) + M [> \frac{4}{3} d] \\ &\quad + M'' [< \epsilon] (t_1 - t' - d), \end{aligned}$$

where  $M'$ ,  $M$  and  $M''$  are mean values of  $\phi(t)$  in the respective intervals and where  $[< \epsilon]$  denotes that the quantity that stands within the brackets is less than  $\epsilon$ .

It is seen that by taking  $\epsilon$  sufficiently small that the sign of

the integral  $\int_{t_0}^{t_1} \phi(t) \chi(t) dt$  is determined by that of  $M \frac{4}{3} d$  and

consequently this integral is different from zero.

Instead of the function  $\chi(t)$ , write

$$\psi(t) = \left( \frac{t - t_0}{t' - t_0} \right)^m \left( \frac{t_1 - t}{t_1 - t'} \right)^n \chi(t),$$

where  $m$  and  $n$  are positive integers.

We see that  $\psi(t_0) = 0 = \psi(t_1)$ , and as above, it follows that

$$\int_{t_0}^{t_1} \phi(t) \psi(t) dt \neq 0.$$

Hence, on the supposition that  $\phi(t) \neq 0$  for a point of the curve along which we integrate, it follows that a function  $\psi(t)$  can be found which causes the above integral to be different from zero.

But as this integral was supposed to be zero for all functions  $\psi(t)$ , it follows that we must have  $\phi(t) = 0$  for all values of  $t$  between  $t_0$  and  $t_1$ .

79. In the expression

$$\Delta I = \epsilon \delta I + \frac{\epsilon^2}{1.2} \delta^2 I + \dots,$$

unless  $\delta I$  and  $\epsilon$  always retain the same sign, it is necessary that  $\delta I$  be zero in order that  $\Delta I$  be continuously negative or continuously positive; *i. e.*, in order that the integral  $I$  be a maximum or a minimum (see Art. 26).

Substitute for  $G_1$  and  $G_2$  their values in terms of  $G$  from Art. 77, in the expression for  $\delta I$  of Art. 76, and we have

$$\delta I = \int_{t_0}^{t_1} G(y' \xi - x' \eta) dt + \left[ \xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'} \right]_{t_0}^{t_1}.$$

If we suppose that the points  $P_0$  and  $P_1$  are fixed so that  $\xi = 0 = \eta$  for them, then the boundary terms vanish. Further, since  $y' \xi - x' \eta$  is an arbitrary and continuous function of  $t$ , it follows from the above lemma that, in order for  $\delta I$  to be zero,  $G = 0$  for every point of the curve within the interval  $t_0 \dots t_1$ .  $G$  can not have a finite value different from zero for isolated points on the curve, since this portion of curve must be continuous in order that the integral may have a meaning.

*The differential equation  $G = 0$  of the second order is a necessary condition for a maximum or a minimum value of  $I$ , and*

will afford the required curve if such curve exists. We note that it is independent of the manner of variation, as the quantities  $\xi$  and  $\eta$  do not appear in it.

From the relation (see Art. 77)

$$\delta I = \int_{t_0}^{t_1} (G_1 \xi + G_2 \eta) dt = 0,$$

it follows that

$$\int_{t_0}^{t_1} G_1 \xi dt + \int_{t_0}^{t_1} G_2 \eta dt = 0.$$

Among all possible variations there are those for which  $\eta = 0$ , and consequently

$$\int_{t_0}^{t_1} G_1 \xi dt = 0.$$

As above we have then

$$G_1 = 0,$$

and similarly

$$G_2 = 0.$$

Further, if we multiply  $y'G = G_1$  and  $-x'G = G_2$  respectively by  $y'$  and  $-x'$ , we have by addition

$$(y'^2 + x'^2)G = 0,$$

and as  $x'$  and  $y'$  cannot both vanish simultaneously, it follows that  $G = 0$ . Hence, the equations  $G_1 = 0$ ,  $G_2 = 0$  on the one hand, and  $G = 0$  on the other, are necessary consequences of one another.

The equations  $G_1 = 0 = G_2$  are often more convenient than  $G = 0$ ; especially is this the case if the function  $F$  does not contain *explicitly* one of the two quantities  $x$  and  $y$ . If  $x$ , for instance, is wanting, then  $\frac{\partial F}{\partial x} = 0$ , and from

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial x'} = 0,$$

it follows that  $\frac{\partial F}{\partial x'} = \text{constant}$ .

80. The curvature at any point of a curve is denoted by

$$\kappa = \frac{1}{\rho} = \frac{x'y'' - y'x''}{[x'^2 + y'^2]^{3/2}},$$

and owing to the equation

$$G = 0 = \frac{\partial^2 F}{\partial y \partial x'} - \frac{\partial^2 F}{\partial x \partial y'} - F_1(x'y'' - x''y'),$$

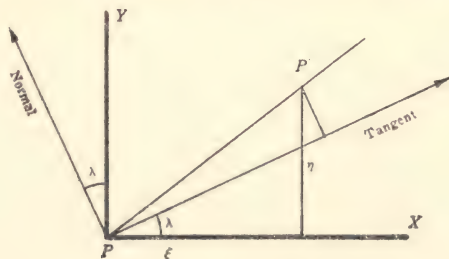
we have

$$\frac{1}{\rho} = \frac{\frac{\partial^2 F}{\partial y \partial x'} - \frac{\partial^2 F}{\partial x \partial y'}}{[x'^2 + y'^2]^{3/2} F_1},$$

an expression which depends upon  $x, y, x', y'$  alone and not upon the higher derivatives.

*It is thus seen that through the equation  $G=0$ , a definite relation is expressed between the curvature of a curve at a point, the coordinates and the direction of the tangent of the curve at this point.*

81. Let a point  $P$  on the curve be transformed by a variation into the point  $P'$ , and let the displacement  $PP'$  be denoted by  $v$ ; further, let the components in the  $x$  and  $y$  directions be  $\xi$  and  $\eta$ , while  $w_N$  and  $w_T$  denote the components of this displacement in the direction of the normal and the tangent to the curve at the point  $P$ .



Let  $\lambda$  denote the angle between these directions and let the direction cosines of the normal be denoted by

$$\frac{\frac{dx}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} \text{ and } \frac{\frac{dy}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}; \text{ i. e., by } \frac{x'}{s'} \text{ and } \frac{y'}{s'}.$$

Then from analytical geometry,

$$w_T = \xi \cos \lambda + \eta \sin \lambda = \frac{x' \xi + y' \eta}{\sqrt{x'^2 + y'^2}},$$

$$w_N = \eta \cos \lambda - \xi \sin \lambda = \frac{x' \eta - y' \xi}{\sqrt{x'^2 + y'^2}},$$

and therefore

$$\xi = \frac{x' w_T - y' w_N}{\sqrt{x'^2 + y'^2}},$$

$$\eta = \frac{y' w_T + x' w_N}{\sqrt{x'^2 + y'^2}},$$

and

$$w_T^2 + w_N^2 = \xi^2 + \eta^2.$$

These expressions substituted in the formula for  $\delta I$  (Art. 79) give

$$\delta I = - \int_{t_0}^{t_1} w_N G ds + \left[ \frac{F w_T}{\sqrt{x'^2 + y'^2}} + \frac{x' \frac{\partial F}{\partial y'} - y' \frac{\partial F}{\partial x'}}{\sqrt{x'^2 + y'^2}} w_N \right]_{t_0}^{t_1}.$$

*From this it is seen that only the component of the variation which is in the direction of the normal enters under the integral sign.*

82. By means of the formula below we will prove that *the variation in the direction of the tangent brings forth only such terms for the first variation that are free from the sign of integration.*

As in Art. 76, write

$$\delta I = \int_{t_0}^{t_1} \left( \frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial x'} \xi' + \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right) dt,$$

and, substituting in this expression  $\xi = v \frac{dx}{ds}$ ,  $\eta = v \frac{dy}{ds}$ , we have

$$\delta I = \int_{t_0}^{t_1} \left\{ v \left( \frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} \right) + \frac{\partial F}{\partial x'} \frac{d}{dt} \left( v \frac{dx}{ds} \right) + \frac{\partial F}{\partial y'} \frac{d}{dt} \left( v \frac{dy}{ds} \right) \right\} dt.$$

Noting that

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'} \frac{d}{dt} \left( v \frac{dx}{ds} \right) = \left[ \frac{\partial F}{\partial x'} v \frac{dx}{ds} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} v \frac{dx}{ds} \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) dt,$$

it is seen that

$$\begin{aligned} \delta I &= \int_{t_0}^{t_1} \left\{ v \frac{dx}{ds} \left[ \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) \right] + v \frac{dy}{ds} \left[ \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right) \right] \right\} dt \\ &\quad + \left[ \frac{\partial F}{\partial x'} v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} G (y' \xi - x' \eta) dt + \left[ \frac{\partial F}{\partial x'} v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1}. \end{aligned}$$

But

$$y' \xi - x' \eta = y' v \frac{dx}{ds} - x' v \frac{dy}{ds} = \frac{v}{ds} \left[ y' \frac{dx}{dt} - x' \frac{dy}{dt} \right] dt \equiv 0,$$

so that everything under the sign of integration drops out, leaving

$$\delta I = \left[ \frac{\partial F}{\partial x'} v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1}.$$

Hence, if we make a sliding of the curve by the substitution

$$\begin{array}{l|l} x & x + \epsilon \xi, \\ y & y + \epsilon \eta, \end{array}$$

and resolve this sliding into two components, of which the one is parallel to the direction of the tangent and the other is parallel to the direction of the normal, then the result of the sliding in the direction of the tangent is seen only in the terms which have reference to the limits, and all these terms are exact differentials under the sign of integration, while the effect due to a sliding in the direction of the normal is shown in the formula of the preceding article.

83. The expression for the first variation has been obtained

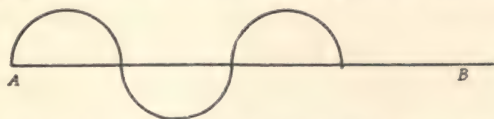
on the hypothesis that the elements of integration have for each point of the path of integration a one-valued meaning. In case the path involved discontinuities, it could be resolved into a finite number of portions of regular curve, and along each portion  $\delta I$  would have a meaning similar to that of the preceding article. It is assumed, then, that the required curve, which is to furnish a maximum or a minimum value of the integral, is regular in its whole trace, or, at least, that it consists of regular portions of curve. In the latter case we shall at first limit ourselves to the consideration of one such portion. Within this portion of curve not only  $x, y$  but also  $x', y'$  will be one-valued functions of  $t$ . This assumption is already implicitly contained in the assumption of the possibility of the development of  $\Delta I$  by Taylor's Theorem; for otherwise the derivatives of  $F$  according to  $x'$  and  $y'$  for the curve which has to be developed could not be formed.

That these assumptions have been made is due to the fact that otherwise the curve could not be the object of a mathematical investigation, since there are no methods of representing irregular curves in their entire generality. If therefore one is contented with the rules of differentiation and integration, he must extend these considerations over only such functions to which the rules may be applied, that is, to the functions having the properties above. There are many problems in geometry and mechanics for which the above hypotheses cannot be made.

84. The following problem proposed by Euler illustrates what has just been said:

*Required a curve connecting two fixed points such that the area between the curve, its evolute and the radii of curvature at its extremities may be a minimum.*

The analytical solution of this problem is the arc of a cycloid, if indeed there exists a minimum. We shall now show that such is not the case. For join the two fixed points  $A$  and  $B$  by a straight line which divide into  $n$  equal parts, and draw alternately



above and below the line  $n$  semi-circles having the  $n$  parts of the line as diameters.

All the radii of curvature of each semi-circle, *i. e.*, of each portion of curve which is to be a minimum, intersect on the line  $AB$  and it is evident that

$$\frac{n}{2} \pi \left( \frac{AB}{2n} \right)^2 = \frac{\pi \overline{AB}^2}{8n}$$

must be a minimum.

If we increase the number  $n$  sufficiently, we may make the above expression become arbitrarily small; and in the limit  $n = \infty$  the curve will tend to become the straight line  $AB$ . From this it is evident that there is not present a minimum surface-area.

85. The same result would have been obtained, if instead of the straight line  $AB$  we had taken the arc of a cycloid through these points, and had then drawn a system of small cycloids having their cusps along the large cycloid. (See Todhunter, *Researches in the Calculus of Variations*, p. 252.) The reason that a minimum is not given through the large cycloid is due to the fact that such a minimum is offered by an irregular curve, and that this irregular curve is not included in our analytical research.\* It follows that our assumption made regarding the regularity of the curve is out of place and leads to something untrue.

But in spite of the not improbable possibility that the curve which is to satisfy a given proposition is irregular, we must make the hypothesis that the curve is regular, since we come to analytical differential equations only by limiting our investigations to such regular curves, and the most general theory of functions teaches that in turn through these differential equations are defined the analytical functions which in their whole extent have existing derivatives.

86. To avoid any misunderstanding, we repeat what we have already said in the previous Chapter: it is not asserted that there is anything in the nature of the problem whereby one may *à priori* conclude that the required curve must be regular. Having these hypotheses, we fix our ideas and draw deductions. After the solution of the problem has been effected, we have to make in

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\* See also Moigno et Lindelöf, *Calcul de Variations*, p. 252.

addition a special proof that the derived curve has all the required properties, and that this curve is the only one which has them.

The chief difficulty in all such problems, as we have shown above in the special problem of approximation (or of the passing to a limit), consists in showing that the regular curve that has been found, found indeed from the necessary conditions, also at the same time satisfies the sufficient conditions, and therefore satisfies *all* the requirements of the problem.

## CHAPTER VI.

THE FORM OF THE SOLUTIONS OF THE DIFFERENTIAL  
EQUATION  $G=0$ .

87. Before we proceed to the development of further conditions for the existence of a maximum or a minimum of the integral

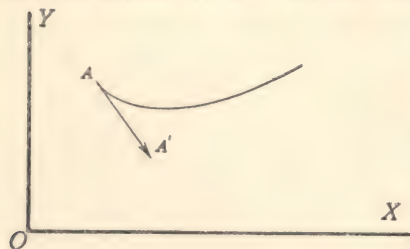
$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt, \quad [1]$$

we shall endeavor to investigate more closely the nature and form of the differential equation  $G=0$ .

We assume that a curve satisfying the differential equation

$$G = \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'} + F_1 \left( x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) = 0, \quad [2]$$

for which  $F_1$  is different from zero, is known.] Let  $A$  be the initial point of the curve, and let  $AA'$  be the direction of the curve



at  $A$ . We suppose that the differential equation  $G=0$  takes its simplest form, if we regard one of the coordinates as a one-valued function of the other. In the integral  $I$  above, the dependence of the quantities  $x, y$  upon the quantity  $t$  is subject only to the

condition that a point is to traverse the curve from the beginning-point to the end-point, when  $t$ , continuously increasing, goes from  $t_0$  to  $t_1$ .

In an infinite number of ways we may introduce in the place of  $t$  another variable  $\tau$ , where

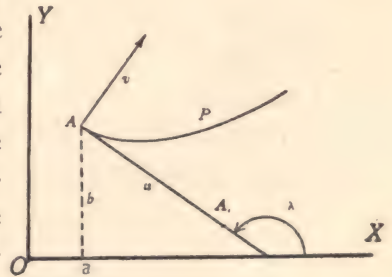
$$t = f(\tau).$$

It is only necessary that the function  $f$  be so formed, that with increasing  $\tau$  also  $t$  increases. In place of  $\tau$  we may reciprocally introduce again

$$\tau = \phi(t).$$

The form of the integral  $I$  and of the differential equation  $G=0$  does not change with these transformations.

Under certain conditions we may choose for  $t$  the coordinate  $x$  itself as independent variable, this being the case when on traversing the curve from the initial-point to the end-point,  $x$  continuously increases. In particular, we may take as the  $X$ -axis the tangent at the initial point  $A$  and take the direction of the curve as the positive direction of the  $X$ -axis. Since we consider only regular curves or curves composed of regular portions, it follows, if the point  $P$  traverses the curve starting from the point  $A$ , that its distance from the normal at  $A$  continuously increases for a certain portion of curve. Hence for this portion of curve, if we take the positive direction of the normal at  $A$  as the positive  $Y$ -axis, there is only one value of  $y$  for every value of  $x$ . Consequently for a definite portion of curve we may always assume that, by a suitable choice of the system of coordinates, the second coordinate may be regarded as a one-valued function of the first. We have therefore only to make a transformation of coordinates.



Let the coordinates of the new origin of coordinates be  $a, b$ , and further let

$$x = a + gu + g'v, \quad y = b + hu + h'v, \quad [3]$$

where  $u$  and  $v$  are the new coordinates.

If  $\lambda$  is the angle between the  $X$ -axis and the  $u$ -axis, we have the well-known relations

$$g = \cos \lambda, \quad h = \sin \lambda, \quad g' = -\sin \lambda, \quad h' = \cos \lambda. \quad [4]$$

The integral  $I$  becomes then, since  $u$  may be regarded as the independent variable,

$$I = \int_0^u F \left( a + gu + g'v, b + hu + h'v, g + g' \frac{dv}{du}, h + h' \frac{dv}{du} \right) du,$$

which we shall for brevity denote by

$$I = \int_0^u f \left( u, v, \frac{dv}{du} \right) du. \quad [5]$$

If we further write  $\frac{dv}{du} = v'$ , then [5] becomes

$$I = \int_0^u f(u, v, v') du. \quad [5^a]$$

We have a differential equation to determine  $v$  as a function of  $u$ , if we apply to this integral the same methods as were used in Arts. 74–80 of the last Chapter.

Let the curve be subjected to a sliding in the direction only of the ordinate  $v$ , and therefore write  $v + \bar{v}$  in the place of  $v$ , where  $\bar{v}$  is a very small quantity which vanishes at the end-point and the initial-point of the portion of curve under consideration. The integral which has been subjected to variation is

$$I + \Delta I = \int f \left( u, v + \bar{v}, v' + \frac{d\bar{v}}{du} \right) du.$$

We next develop  $\Delta I$  according to powers of  $\bar{v}$  and  $\frac{d\bar{v}}{du}$ . The aggregate of the terms of the first dimension is

$$\delta I = \int_0^u \left\{ \frac{\partial f}{\partial v} \bar{v} + \frac{\partial f}{\partial v'} \frac{d\bar{v}}{du} \right\} du, \quad [6]$$

or, since

$$\frac{\partial f}{\partial v'} \frac{d\bar{v}}{du} = \frac{d}{du} \left( \bar{v} \frac{\partial f}{\partial v'} \right) - \bar{v} \frac{d}{du} \frac{\partial f}{\partial v'},$$

we have

$$\delta I = \int_0^u \left\{ \frac{\partial f}{\partial v} - \frac{d}{du} \frac{\partial f}{\partial v'} \right\} \bar{v} du + \left[ \bar{v} \frac{df}{dv'} \right]_0^u. \quad [6^a]$$

The quantity in the square brackets vanishes, because  $\bar{v}=0$  on the limits. Further, we must have

$$\delta I = 0.$$

Since  $\bar{v}$  is arbitrary, subjected only to the condition that it must vanish on the limits, it follows from the lemma of the preceding Chapter that

$$\frac{d}{du} \frac{\partial f}{\partial v'} - \frac{\partial f}{\partial v} = 0,$$

or

$$\frac{\partial^2 f}{\partial v'^2} \frac{dv'}{du} + \frac{\partial^2 f}{\partial v \partial v'} v' + \frac{\partial^2 f}{\partial u \partial v'} - \frac{\partial f}{\partial v} = 0. \quad [2^a]$$

88. If one of the coordinates can be regarded as a one-valued function of the other, the equation  $[2^a]$  may take the place of the form  $[2]$  for  $G=0$ .

We shall now show that the quantity  $\frac{\partial^2 f}{\partial v'^2}$  which enters in  $[2^a]$  is identical with  $F_1$ , provided that in the function  $F(x, y, x', y')$   $x$  and  $y$  may be regarded as functions of  $u$  alone.

$$\begin{aligned} \text{Since } f(u, v, v') &\equiv F(a + g u + g' v, b + h u + h' v, g + g' v', h + h' v') \\ &\equiv F(x, y, x', y'), \end{aligned}$$

it follows that

$$\frac{\partial f}{\partial v'} = \frac{\partial F}{\partial x'} g' + \frac{\partial F}{\partial y'} h',$$

and consequently

$$\frac{\partial^2 f}{\partial v'^2} = \frac{\partial^2 F}{\partial x'^2} g'^2 + 2 \frac{\partial^2 F}{\partial x' \partial y'} g' h' + \frac{\partial^2 F}{\partial y'^2} h'^2.$$

On the other hand by its definition  $F_1$  was determined by any of the relations:

$$\frac{\partial^2 F}{\partial x'^2} = F_1 y'^2; \quad \frac{\partial^2 F}{\partial x' \partial y'} = -F_1 x' y'; \quad \frac{\partial^2 F}{\partial y'^2} = F_1 x'^2. \quad [7]$$

From this it follows that

$$\begin{aligned} \frac{\partial^2 f}{\partial v'^2} &= \{g'^2 y'^2 - 2g' h' x' y' + h'^2 x'^2\} F_1 \\ &= \{g' y' - h' x'\}^2 F_1 \\ &= \{g' h - h' g\}^2 F_1 \\ &= \{-\sin^2 \lambda - \cos^2 \lambda\}^2 F_1; \end{aligned}$$

or finally,

$$\frac{\partial^2 f}{\partial v'^2} = F_1. \quad [8]$$

Hence  $[2^a]$  may be written

$$F_1 \frac{dv'}{du} + \frac{\partial^2 f}{\partial v \partial v'} v' + \frac{\partial^2 f}{\partial u \partial v'} - \frac{\partial f}{\partial v} = 0.$$

Since we have

$$v' = \frac{dv}{du},$$

and

$$f(u, v, v') \equiv F(x, y, x', y'),$$

where  $x, y$  are determined in terms of  $u, v$  from  $[3]$ , it follows that

$$F_1 \frac{d^2 v}{du^2} + \frac{\partial^2 F}{\partial v \partial v'} \frac{dv}{du} + \frac{\partial^2 F}{\partial u \partial v'} - \frac{\partial F}{\partial v} = 0. \quad [2^b]$$

89. In the theory of differential equations it is known that every differential equation of the form  $[2^b]$  may be integrated in the form of a power-series of the independent variable  $u$ .

As a special case we have the following:

Suppose 1) that at the initial point of the curve represented by the power-series which is to be formed, we have

$$u = 0, v = b_0,$$

where  $b_0$  is an arbitrary constant;

2) that the direction of the curve at the initial point is determined by the arbitrary constant

$$\frac{dv}{du} = v'_0;$$

then  $v$  for sufficiently small values of  $u$  may be expressed in the power-series

$$v = b_0 + v'_0 u + \dots, \quad [9]$$

where we have assumed that  $F_1$  is different from zero at the initial point  $(0, b_0)$ .

The second and higher derivatives of  $v$  on the position  $(0, b_0, v'_0)$  may all be derived from the differential equation  $[2^b]$ . Hence, in  $[9]$  we have  $v$  as a power-series in  $u$ , whose coefficients contain, besides the constants had in each problem, only the two arbitrary constants  $b_0$  and  $v'_0$ , which change from curve to curve.

90. If we substitute the expression for  $v$  given by equation  $[9]$  in the formulæ  $[3]$ , we have  $x$  and  $y$  expressed in terms of  $u$ . In these expressions there appear the constants  $g, g', h, h'$ , which depend upon  $\lambda$  and also upon the coordinates  $a, b$  of the origin of the  $u, v$  system of coordinates, and the two constants of integration  $b_0$  and  $v'_0$ , defined in Art. 89. These latter constants vary from curve to curve. In these formulæ, just as in Art. 89, we can ascribe only small values to the quantity  $u$ .

We know, however, as is seen in the theory of functions, that if a curve is given only in a small portion, its continuation is thereby completely determined. We therefore need to know the curve only for indefinitely small  $u$ 's in order to be able to follow its trace at pleasure.

The coordinates  $x, y$  of the curve may be represented as functions of  $t$  and two arbitrary constants  $\alpha$  and  $\beta$ . Instead of  $u$ , we may introduce an arbitrary function of another quantity, if only this quantity increases in a continuous manner, when the curve is traversed from the beginning-point to the end-point. As already mentioned, the two constants of integration vary from curve to curve. If we determine suitably these constants, we can compel the curve, which satisfies the differential equation  $G=0$ , to pass through two prescribed points.

In this manner we have a clear representation of the manner in which the analytical expressions giving  $x$  and  $y$  are derived;  $x$  and  $y$  are found in general from the equation  $G=0$  in the form

$$x=\phi(t, \alpha, \beta), \quad y=\psi(t, \alpha, \beta). \quad [10]$$

At the same time it is seen that up to a certain point, at least,  $x$  and  $y$  are one-valued and regular functions of  $t$  and of the two constants of integration  $\alpha$  and  $\beta$ , so that eventually we can also differentiate with respect to these two constants.

91. It seems desirable here in connection with what was given in Arts. 89, 90 to consider the exceptional case, *viz.*, the one in which

$$\frac{\partial^2 f}{\partial v'^2} = F_1 = \frac{\partial^2 F}{\partial x'^2} \sin^2 \lambda - 2 \frac{\partial^2 F}{\partial x' \partial y'} \sin \lambda \cos \lambda + \frac{\partial^2 F}{\partial y'^2} \cos^2 \lambda \quad [8]$$

is equal to zero for the origin ( $u=0, v=b_0$ ) of the curve which satisfies the equation  $G=0$ .

We shall see that this is only an exceptional case by showing the following :

If we draw around the point ( $u=0, v=b_0$ ) a small circle, then this circle may be so distributed into sectors that within each sector  $\frac{\partial^2 f}{\partial v'^2}$  is not equal to zero. For we may regard the radius of a sufficiently small circle about the initial point ( $u=0, v=b_0$ ) of the curve in question as the initial direction of this curve. If for  $v$  we write in [8] the power-series given in [9], we have, by putting  $F_1=0$ , an equation for the determination of  $v_0'$ , that is, the quantity which fixes the initial direction. This equation has either no real roots, and then there will exist no curve starting from the point ( $u=0, v=b_0$ ), or  $F_1$  vanishes for single  $v_0'$ 's, and then the radii determine separate sectors. Within these sectors curves may be drawn starting from the point ( $u=0, v=b_0$ ) in every direction, for which  $F_1$  is different from zero. Consequently one can always assign limits for  $v_0'$  within which the corresponding curves, satisfying the equation  $G=0$  and starting from the point ( $u=0, v=b_0$ ), have at the origin, at least, an  $F_1$  different from zero.

92. Finally we shall show that the curves starting from the same point ( $u=0, v=b_0$ ) which satisfy the equation  $G=0$  lie completely separated from one another at their initial point.

If we draw a small circle around the point ( $u=0, v=b_0$ ), then on its periphery we can easily determine the point  $u, v$  in which it is cut by one of the curves in question. For let  $\rho$  be the radius of the small circle, so that

$$u^2 + (v - b_0)^2 = \rho^2. \quad [11]$$

Writing for  $v$  the power-series [9], we have

$$\begin{aligned} \rho^2 &= (1 + v_0'^2)u^2 + C_3 u^3 + \dots, \\ \text{or} \quad \rho &= \sqrt{1 + v_0'^2} u + (u)_2 + \dots \end{aligned}$$

We may revert this series and have  $u$  expressed as a function of  $\rho$ , so that

$$\begin{aligned} u &= \frac{1}{\sqrt{1 + v_0'^2}} \rho + (\rho)_2' + \dots, \\ \text{and therefore} \quad v - b_0 &= \frac{v_0'}{\sqrt{1 + v_0'^2}} \rho + (\rho)_2'' + \dots \end{aligned} \quad [12]$$

These series are convergent for all  $\rho$ 's within a certain limit  $\rho_0$ , so that therefore  $u$  and  $v$ , the coordinates of the point to be determined upon the periphery of the circle with the radius  $\rho$ , are uniquely found for all values of  $\rho < \rho_0$ . Consequently at the beginning, at least, the curves which belong to a sector in reality lie completely separated from one another.

93. *The form of the differential equation  $G = 0$ , where  $s$  is introduced as the independent variable instead of  $t$ .* If we introduce instead of  $t$  another variable  $\tau$  by writing  $t$  equal to a function of  $\tau$ , we arrive at the same differential equation  $G = 0$ . It is usually advantageous to introduce the length of arc  $s$  as the independent variable.

Since (Art. 68) the derivatives of the function  $F$  with respect to its third and fourth arguments are invariative, we have (writing  $k = \frac{1}{\sqrt{x'^2 + y'^2}} = \frac{dt}{ds}$  in the formulæ of Art. 68)

$$\frac{\partial}{\partial x'} F(x, y, x', y') = \frac{\partial}{\partial \left(\frac{dx}{ds}\right)} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right),$$

$$\frac{\partial}{\partial y'} F(x, y, x', y') = \frac{\partial}{\partial \left(\frac{dy}{ds}\right)} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right).$$

From this it is seen that  $\frac{\partial F}{\partial x'}$ ,  $\frac{\partial F}{\partial y'}$  are independent of the manner in which  $x$  and  $y$  are expressed as functions of  $t$ , and depend only upon the point in question of the curve and the direction of the tangent at this point. We have at once

$$\frac{\partial^2 F}{\partial x' \partial y'} = \frac{\partial^2}{\partial \frac{dx}{ds} \partial \frac{dy}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) \frac{1}{\frac{ds}{dt}};$$

and since

$$F_1 = -\frac{1}{x'y'} \frac{\partial^2 F(x, y, x', y')}{\partial x' \partial y'},$$

it follows that

$$F_1 \left(\frac{ds}{dt}\right)^3 = -\frac{1}{\frac{dx}{ds} \frac{dy}{ds}} \cdot \frac{\partial^2}{\partial \frac{dx}{ds} \partial \frac{dy}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right).$$

If further we write

$$\frac{\partial^2}{\partial \frac{dx}{ds} \partial \frac{dy}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) = -\frac{dx}{ds} \frac{dy}{ds} F_1\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right),$$

we have

$$F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) \left(\frac{ds}{dt}\right)^3 = F_1\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right).$$

Hence the equation  $G=0$  becomes

$$\begin{aligned} & \frac{\partial^2}{\partial x \partial \frac{dy}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) - \frac{\partial^2}{\partial y \partial \frac{dx}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) \\ & + \left(\frac{dx}{ds} \frac{d^2 y}{ds^2} - \frac{dy}{ds} \frac{d^2 x}{ds^2}\right) F_1\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) = 0. \end{aligned}$$

94. From the above equation the second and all higher derivatives of  $x$  and  $y$  with respect to  $s$  may be explicitly expressed in terms of  $x, y, \frac{dx}{ds}, \frac{dy}{ds}$ .

For, from the relation

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

it follows, through differentiation, that

$$\frac{dx}{ds} \frac{d^2x}{ds^2} + \frac{dy}{ds} \frac{d^2y}{ds^2} = 0. \quad [i]$$

If, for brevity, we write

$$\begin{aligned} & \frac{\partial^2}{\partial x \partial \frac{dy}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) - \frac{\partial^2}{\partial y \partial \frac{dx}{ds}} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) \\ & = H\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right), \end{aligned}$$

we may write the differential equation of the last article in the form:

$$F_1\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) \left(\frac{dy}{ds} \frac{d^2x}{ds^2} - \frac{dx}{ds} \frac{d^2y}{ds^2}\right) = H\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right).$$

With the aid of [i], we have

$$\left. \begin{aligned} \frac{d^2x}{ds^2} F_1\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) &= \frac{dy}{ds} H\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right), \\ \frac{d^2y}{ds^2} F_1\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) &= -\frac{dx}{ds} H\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right). \end{aligned} \right\} \quad [ii]$$

It requires no further explanation to show how from these relations one can express the third and higher derivatives of  $x$  and  $y$  with respect to  $s$  in terms of  $x, y, \frac{dx}{ds}$  and  $\frac{dy}{ds}$ .

95. If, then,  $F_1$  nowhere vanishes, and like  $H$  is a continuous

function of its arguments, and if  $H$  never becomes infinite (see Art. 149), it follows that  $\frac{d^2x}{ds^2}$  and  $\frac{d^2y}{ds^2}$  can never become infinitely large, and are also continuous functions of the arc.

It follows that the curve has no singular point within the interval in question and that the curvature nowhere becomes infinitely large. This may be shown in the following manner: Let the points  $x_0, y_0$  of the curve correspond to the value  $s_0$  of  $s$ , then owing to the equations [ii] of the preceding article, the curve in the neighborhood of this point may be represented by the equations

$$x = x_0 + A(s - s_0)^\mu + \dots,$$

$$y = y_0 + B(s - s_0)^\mu + \dots,$$

where the constants  $A$  and  $B$  do not vanish simultaneously. When the values of  $\frac{dx}{ds}$  and  $\frac{dy}{ds}$  derived from these equations are substituted in

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

it follows that

$$1 = \mu^2(A^2 + B^2)(s - s_0)^{2\mu-2} + A_1(s - s_0)^{2\mu-3} + \dots,$$

and since this equation is true for all points in the neighborhood of  $x_0, y_0$ , it is seen that

$$A_1 = A_2 = \dots = 0,$$

and that further

$$\mu = 1, A^2 + B^2 = 1.$$

Hence the coordinates of every point of the curve situated in the neighborhood of  $x_0, y_0$  may be represented through the regular functions

$$x = x_0 + A(s - s_0) + \dots,$$

$$y = y_0 + B(s - s_0) + \dots,$$

where  $A$  and  $B$  do not simultaneously vanish. Since this is true for every point  $x_0, y_0$ , it follows that the curve can have no singular points. Hence also the quantities  $x'$  and  $y'$  can not both vanish at the same time.

## CHAPTER VII.

REMOVAL OF CERTAIN LIMITATIONS THAT HAVE BEEN MADE.  
 INTEGRATION OF THE DIFFERENTIAL EQUATION  $G=0$   
 FOR THE PROBLEMS OF CHAPTER I.

96. In the derivation of the formulæ of Chapter V, it was presupposed that the portion of curve under consideration changed its direction in a continuous manner throughout its whole trace; that is,  $x'$ ,  $y'$  varied in a continuous manner. We shall now assume only that the curve is composed of regular portions of curve; so that, therefore, the tangent need not vary continuously at every point of the curve. Then it may be shown as follows that each portion of curve must satisfy the differential equation  $G=0$ . For if the curve consists of two regular portions  $AC$  and  $CB$ , then among all possible variations of  $AB$  there exist those in which  $CB$  remains unchanged, and only  $AC$  is subjected to variation.

As above, we conclude that this portion of curve must satisfy the differential equation  $G=0$ . The same is true of  $CB$ .

We may now do away with the restriction that the curve consists of one regular trace, and assume that it consists of a finite number of regular traces.

97. Suppose that the function  $F$  does not contain explicitly the variable  $x$ , and consequently  $\frac{\partial F}{\partial x}=0$ . Instead of the equation  $G=0$ , let us take  $G_1=0$ , or

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial x'} = 0.$$

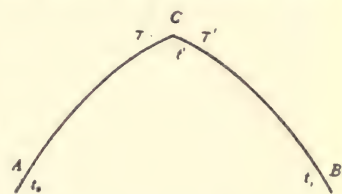
It follows that

$$\frac{\partial F}{\partial x'} = \text{constant},$$

the constant being independent of  $t$ ; *à priori*, however, we do not know that  $\frac{\partial F}{\partial x'}$  does not undergo a sudden change at points of discontinuity of  $x'$  and  $y'$ . Consequently, the more important is the following theorem for the integration of the differential equation  $G=0$ :

*Even if  $x'$ ,  $y'$ , and thereby also the direction of the curve, suffer at certain points sudden changes, nevertheless, the quantities  $\frac{\partial F}{\partial x'}$ ,  $\frac{\partial F}{\partial y'}$  vary in a continuous manner throughout the whole curve for which  $G=0$ .*

If  $t'$  is a point of discontinuity in the curve, then on both sides of  $t'$  we take the points  $\tau$  and  $\tau'$  in such a manner that within the portions  $\tau \dots t'$  and  $t' \dots \tau$  there is no other discontinuity in the direction of the curve. Then a possible variation of the curve is also the one by which  $t_0 \dots \tau$  and  $\tau' \dots t_1$  remain unaltered and only the portion  $\tau \dots \tau'$  is varied. Here the points  $\tau$  and  $\tau'$  are supposed to remain fixed, while  $t'$  is subjected to any kind of sliding.



The variation of the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt,$$

then depends only upon the variations of the sum of the integrals

$$\int_{\tau}^{t'} F(x, y, x', y') dt + \int_{t'}^{\tau'} F(x, y, x', y') dt.$$

Since the first variation of this expression must vanish, we necessarily have (Art. 79)

$$0 = \int_{\tau}^{t'} G(y' \xi - x' \eta) dt + \int_{t'}^{\tau'} G(y' \xi - x' \eta) dt \\ + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{\tau}^{t'} + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t'}^{\tau'}.$$

Since  $G=0$  along the whole curve, it follows that

$$\left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{\tau}^{t'} + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t'}^{\tau'} = 0.$$

The quantities  $\xi$  and  $\eta$  are both zero at the fixed points  $\tau$  and  $\tau'$ ; and, if we denote the values that may belong to the quantity  $\frac{\partial F}{\partial x'}$ , according as we approach the point  $t'$  from the points  $\tau$  or  $\tau'$  by

$$\left[ \frac{\partial F}{\partial x'} \right]_{t'}^{-} \text{ and } \left[ \frac{\partial F}{\partial x'} \right]_{t'}^{+},$$

the above expression becomes

$$\left\{ \left[ \frac{\partial F}{\partial x'} \right]_{t'}^{-} - \left[ \frac{\partial F}{\partial x'} \right]_{t'}^{+} \right\} (\xi)' + \left\{ \left[ \frac{\partial F}{\partial y'} \right]_{t'}^{-} - \left[ \frac{\partial F}{\partial y'} \right]_{t'}^{+} \right\} (\eta)' = 0,$$

where  $(\xi)'$  and  $(\eta)'$  are the values of  $\xi$  and  $\eta$  at the point  $t'$ . Since the quantities  $(\xi)'$  and  $(\eta)'$  are quite arbitrary, it follows that their coefficients in the above expression must respectively vanish, so that

$$\left[ \frac{\partial F}{\partial x'} \right]_{t'}^{-} = \left[ \frac{\partial F}{\partial x'} \right]_{t'}^{+} \text{ and } \left[ \frac{\partial F}{\partial y'} \right]_{t'}^{-} = \left[ \frac{\partial F}{\partial y'} \right]_{t'}^{+};$$

that is, *the quantities  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  vary in a continuous manner by the transition from one regular part of the curve to the other, even if  $x'$  and  $y'$  at this point suffer sudden changes.*

This is a new necessary condition for the existence of a maximum or a minimum of the integral  $I$ , which does not depend upon the nature of the differential equation  $G = 0$ .

98. The question naturally arises: *How is it possible that the functions  $\frac{\partial F}{\partial x'}$ ,  $\frac{\partial F}{\partial y'}$ , which depend upon  $x'$  and  $y'$ , vary in a continuous manner, even when  $x'$  and  $y'$  experience discontinuities?* To answer this question we may say that the composition of these functions is of a peculiar nature, viz., the terms which contain  $x'$ ,  $y'$  are multiplied by functions which vanish at the points considered. This is illustrated more clearly in the example treated in Art. 100. The theorem is of the greatest importance in the determination of the constant. In the special case of the preceding article, where  $\frac{\partial F}{\partial x'} = \text{constant}$ , it is clear that this constant must have the same value for all points of the curve. The theorem may also be used in many cases to prove that the direction of the curve nowhere changes in a discontinuous manner, and consequently does not consist of several regular portions but of one single regular trace. This is also illustrated in the examples which follow (Arts. 100 *et seq.*).

99. We may give here a summary of what has been obtained through the vanishing of the first variation as necessary conditions for the existence of a maximum or a minimum of the integral  $I$ :

1) *The curve offering the maximum or minimum must satisfy the differential equation*

$$G \equiv \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'} + F_1 \left( x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) = 0,$$

*or, what is the same thing, the two equations*

$$G_2 \equiv \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) = 0, \quad G_2 \equiv \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right) = 0;$$

2) *The two derivatives of the function  $F$  with respect to  $x'$  and  $y'$  must vary in a continuous manner even at the points where the direction of the curve does not vary continuously.*

In order to establish the criteria by means of which it may be ascertained whether the curve determined through the equation  $G=0$  offers a maximum or a minimum, we must investigate the terms of the second dimension in  $\Delta I$  of Chapter V. First, however, to make clear what has already been written, we may apply our deductions to some of the problems already proposed.

SOLUTION OF THE DIFFERENTIAL EQUATION  $G=0$  FOR THE PROBLEMS OF CHAPTER I.

100. Let us consider Problem I of Art. 7. The integral which we have to minimize is

$$\frac{S}{2\pi} = \int_{t_0}^{t_1} y \sqrt{x'^2 + y'^2} dt. \quad [1]$$

Hence

$$F = y \sqrt{x'^2 + y'^2}, \quad [2]$$

and consequently

$$\frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{x'^2 + y'^2}}; \quad \frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{x'^2 + y'^2}}. \quad [3]$$

From this it is seen that  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  are proportional to the direction cosines of the tangent to the curve at any point  $x(t), y(t)$ ; and, since  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  must vary everywhere in a continuous manner, it follows also that the direction of the curve varies everywhere in a continuous manner except for the case where  $y=0$ . But the quantity  $\sqrt{x'^2 + y'^2}$  varies in a discontinuous manner if  $x'$  and  $y'$  are discontinuous; at the same time, however,  $y$  is equal to zero, as is more clearly seen in the figure below.

Since  $F$  does not contain  $x$  *explicitly*, we may use the equation

$$G_1 = 0, \quad \text{or} \quad \frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{x'^2 + y'^2}} = \beta, \quad [4]$$

where  $\beta$  is the constant of integration. Hence

$$y^2 \left( \frac{dx}{dt} \right)^2 = \beta^2 \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right]. \quad [5]$$

The solution of this equation is the catenary :

$$\left. \begin{aligned} x &= a + \beta t, \\ y &= \beta/2 (e^t + e^{-t}), \end{aligned} \right\} \quad [6]$$

where  $a$  is a second arbitrary constant.

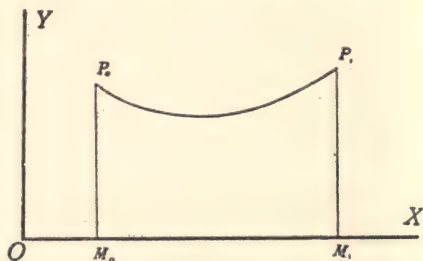
101. *A discontinuous solution.* If we take the arc  $s$  as independent variable instead of the variable  $t$ , the differential equation of the curve is

$$y \frac{dx}{ds} = \beta.$$

Suppose that  $\beta = 0$ , which value it must retain within the whole interval  $t_0 \dots t_1$ . Further, since  $y \neq 0$  at the point  $P_0$ , it follows that  $\frac{dx}{ds} = \cos \phi = 0$  (where  $\phi$  is the angle which the tangent makes with the  $X$ -axis), and that  $\cos \phi$  must remain zero until  $y = 0$ ; that is, the point which describes the curve must move along the ordinate  $P_0M_0$  to the point

$M_0$ . At this point  $\frac{dx}{ds}$  cannot, and

must not, equal zero if the point is to move to  $P_1$ . Hence, at  $M_0$  there is a sudden change in the direction of the curve, as there is again at the point  $M_1$ . The curve giving



the minimum surface of revolution is consequently, in this case, offered by the irregular trace  $P_0M_0M_1P_1$ . The case where  $\beta = 0$  may be regarded as an exceptional case. The unconstrained lines  $P_0M_0$  and  $P_1M_1$ , i. e.,  $x = x_0$  and  $x = x_1$  satisfy the condition  $G = 0$ , since  $y'G = G_1$ , and for these values  $G_1 = 0$ ; also for these lines.  $y \neq 0$ . But  $G \neq 0$  for the restricted portion  $M_0M_1$  and is, in fact equal to 1.

102. We may prove as follows that the two ordinates and the section of the  $X$ -axis give a minimum. This is seen at once when we have shown that the first variation for all allowable deformations is *positive*. The problem is a particular case of Art. 79.

The first variation may be decomposed into several parts (cf. Arts. 79 and 81):

$$\begin{aligned}\delta I = & - \int_{\bar{M}_0}^{P_0} G w_N ds + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{M_0}^{P_0} \\ & - \int_{\bar{M}_1}^{M_0} G w_N ds + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{M_1}^{M_0} \\ & - \int_{P_1}^{M_1} G w_N ds + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{P_1}^{M_1}.\end{aligned}$$

Now all the boundary terms are zero, since

$$\frac{\partial F}{\partial x'} = \frac{y x'}{\sqrt{x'^2 + y'^2}}, \quad \frac{\partial F}{\partial y'} = \frac{y y'}{\sqrt{x'^2 + y'^2}},$$

and therefore both are zero at the points  $M_0$  and  $M_1$ , while  $\xi$  and  $\eta$  are zero at  $P_0$  and  $P_1$ . In the first and third integrals  $G=0$ ; in the second this function equals unity, and if we reverse the limits,  $ds$  is positive, as is also  $w_N$ . Hence, the first variation  $\delta I$  is always *positive*.

When the arbitrary constant  $\beta \neq 0$ , the curve consists of one regular trace that lies wholly above the  $X$ -axis. Further investigation is necessary to determine when this curve offers in reality a minimum.

103. In the second problem (Art. 9), we have for the time of falling the integral

$$T = \int_{t_0}^{t_1} \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + a^2}} dt. \quad [1]$$

That this expression may, in reality, express the time of falling (the time and, therefore, also the increment  $dt$  being essentially a positive quantity), the two roots that appear under the integral sign must always have the same sign. Since  $\sqrt{4gy + a^2}$  can

always be chosen positive, it follows that  $\sqrt{x'^2 + y'^2}$  must be positive within the interval  $t_0 \dots t_1$ .

It might happen, however, if we express  $x$  and  $y$  in terms of  $t$ , that  $x'$  and  $y'$  might both vanish for a value of  $t$  within the interval  $t_0 \dots t_1$ . In this case the curve has at the point  $x, y$ , which belongs to this value of  $t$ , a singular point, at which the velocity of the moving point is zero.

Suppose that this is the case for  $t=t'$ , and that the corresponding point is  $x_0, y_0$ , so that we have

$$x = x_0 + a(t - t')^m + \dots,$$

$$y = y_0 + b(t - t')^m + \dots,$$

where  $m \geq 2$ , and at least one of the two quantities  $a$  and  $b$  is different from zero.

Then is

$$x'^2 + y'^2 = m^2 (a^2 + b^2) (t - t')^{2(m-1)} + \dots,$$

and

$$\sqrt{x'^2 + y'^2} = m \sqrt{a^2 + b^2} (t - t')^{m-1} + \dots$$

Here we may suppose  $\sqrt{a^2 + b^2}$  positive.

If now  $m$  is odd, then for small values of  $t - t'$ , the expression on the right is positive, and hence  $\sqrt{x'^2 + y'^2}$  always has a positive sign.

If on the contrary  $m$  is even, equal to 2, say, then the curve has at the point  $x_0, y_0$  a *cusp*, since here  $\sqrt{x'^2 + y'^2}$  has a positive or a negative value according as  $t > t'$  or  $t < t'$ .

If therefore the above integral is to express the time,  $\sqrt{x'^2 + y'^2}$  cannot always be put equal to the same series of  $t$ , but must after passing the *cusp* be put equal to the opposite value of the series. We therefore limit ourselves to the consideration of a portion of the curve which is free from singular points.

Such limitations must often be made in problems, since otherwise the integrals have no definite meaning. Hence with this supposition  $\sqrt{x'^2 + y'^2}$  will never equal zero.

We may then write :

$$F = \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + a^2}}, \quad [2]$$

and consequently

$$\left. \begin{aligned} \frac{\partial F}{\partial x'} &= \frac{1}{\sqrt{4gy + a^2}} \frac{x'}{\sqrt{x'^2 + y'^2}}, \\ \frac{\partial F}{\partial y'} &= \frac{1}{\sqrt{4gy + a^2}} \frac{y'}{\sqrt{x'^2 + y'^2}}. \end{aligned} \right\} \quad [3]$$

From this we may conclude, in a similar manner as in the first example, that  $\frac{\partial F}{\partial x'}$ ,  $\frac{\partial F}{\partial y'}$  are proportional to the direction cosines of the tangent of the curve at the point  $x, y$ . Since now  $\frac{\partial F}{\partial x'}$ ,  $\frac{\partial F}{\partial y'}$  vary in a continuous manner along the whole curve, and since, further,  $\sqrt{4gy + a^2}$  has a definite value which is different from zero, it follows also that the direction of the required curve varies in a continuous manner, or the curve must consist of one single trace.

Also here  $F$  is independent of  $x$ , and consequently we employ the differential equation  $G_1 = 0$ , from which we have

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{4gy + a^2}} \frac{x'}{\sqrt{x'^2 + y'^2}} = C, \quad [4]$$

where  $C$  is an arbitrary constant.

If  $C$  is equal to zero, then in the whole extent of the curve  $C$  must equal zero; and consequently, since  $\sqrt{4gy + a^2}$  is neither 0 nor  $\infty$ ,  $\frac{x'}{\sqrt{x'^2 + y'^2}} = \cos \alpha$  must always equal zero; that is, the curve must be a vertical line. Neglecting this self-evident case,  $C$  must have a definite value which is always the same for the whole curve and different from zero.

From [4], it follows that

$$dx^2 = C^2(4gy + a^2) (dx^2 + dy^2),$$

or, if we absorb  $4g$  in the arbitrary constant and write

$$\frac{a^2}{4g} = a, \quad \text{and} \quad 4gC^2 = c^2,$$

we have

$$dx^2 = c^2(y + a) (dx^2 + dy^2);$$

whence

$$dx = \frac{c(y + a) dy}{\sqrt{(y + a) [1 - c^2(y + a)]}}. \quad [5]$$

In order to perform this last integration, write

$$d\phi = \frac{c dy}{\sqrt{(y+a)[1-c^2(y+a)]}}; \quad [6]$$

therefore

$$dx = (y+a) d\phi. \quad [5^a]$$

In the expression for  $d\phi$ , write

$$2c^2(y+a) = 1-\xi. \quad [7]$$

Then is

$$2[1-c^2(y+a)] = 1+\xi, \quad [8]$$

and

$$2c^2 dy = -d\xi. \quad [9]$$

Therefore

$$d\phi = -\frac{d\xi}{\sqrt{1-\xi^2}}, \quad [10]$$

and hence

$$\xi = \cos \phi. \quad [11]$$

Here the constant of integration may be omitted, since  $\phi$  itself is fully arbitrary.

Hence,

$$\left. \begin{aligned} y+a &= \frac{1}{2c^2}(1-\cos \phi), \\ x+x_0 &= \frac{1}{2c^2}(\phi-\sin \phi); \end{aligned} \right\} \quad [12]$$

equations, which represent a *cycloid*.

The constants of integration  $x_0$ ,  $c$  are determined from the condition that the curve is to go through the two points  $A$  and  $B$ . Now develop  $x$  and  $y$  in powers of  $\phi$ : then in  $y$  the lowest power is  $\phi^2$ , and in  $x$  it is  $\phi$ ; so that the curve has in reality a cusp for  $\phi=0$ , and this is repeated for  $\phi=2\pi, 4\pi, \dots$

$A$  and  $B$  must lie between two consecutive cusps (Art. 104).

The curve may be constructed, if we draw a horizontal line through the point  $-x_0, -a$ , and construct on the under side of this line a circle with radius  $1/(2c^2)$ , which touches the horizontal line at the point  $-x_0, -a$ . Let this circle roll in the positive

$X$ -direction on the horizontal line, then the original point of contact describes a cycloid which goes through  $A$  and  $B$  and which satisfies the differential equation.

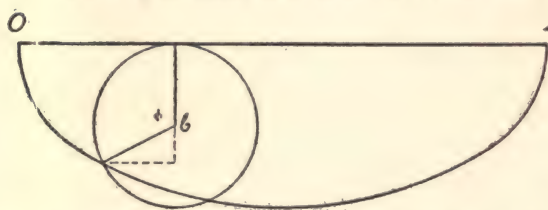
104. That the points  $A$  and  $B$  cannot lie upon different loops of a cycloid may be seen as follows: For simplicity, let the initial velocity  $a$  be zero and shift the origin of coordinates so as to get rid of the constants.

The equation of the cycloid is then

$$\left. \begin{aligned} x &= r(\phi - \sin \phi), \\ y &= r(1 - \cos \phi), \end{aligned} \right\}$$

where we have written  $r$  in the place of  $1/(2c^2)$ .

The cycloidal arc is seen from the accompanying figure. Take



$x$  two points lying upon different loops very near and symmetrically situated with respect to an apex, and let us compare the time it would take to

travel from one of these points to the other by the way of the apex with the time taken over a straight line joining them. The parameters of the two points may be expressed by

$$\phi_0 = 2\pi - \psi_0,$$

$$\phi_1 = 2\pi + \psi_0.$$

The time required to go by the way of the apex is

$$T = \frac{1}{\sqrt{2g}} \int_{t_0}^{t_1} \sqrt{\frac{x'^2 + y'^2}{y}} dt = \frac{1}{\sqrt{2g}} \int_{s_0}^{s_1} \frac{ds}{\sqrt{y}}.$$

Now

$$dx = r(1 - \cos \phi) d\phi,$$

and

$$dy = r \sin \phi d\phi,$$

so that

$$ds = \sqrt{dx^2 + dy^2} = 2r \sin(\phi/2) d\phi,$$

and consequently

$$T = \frac{1}{\sqrt{2g}} \int_{\phi_0}^{\phi_1} \frac{2r \sin(\phi/2)}{\sqrt{r} \sqrt{1 - \cos \phi}} d\phi = \sqrt{\frac{r}{g}} \int_{\phi_0}^{\phi_1} d\phi$$

$$= \sqrt{\frac{r}{g}} (\phi_1 - \phi_0) = \sqrt{\frac{r}{g}} [2\pi + \psi_0 - 2\pi + \psi_0] = 2 \sqrt{\frac{r}{g}} \psi_0.$$

The component of velocity across the horizontal line from  $\phi_0$  to  $\phi_1$  is  $\left[ v \frac{dx}{ds} \right]$  or, since  $\frac{dx}{ds} = \sin \frac{\phi}{2}$  and  $v^2 = 2gy$ , this component is equal to

$$\left[ \sqrt{2gr} \sqrt{1 - \cos \phi} \sin \frac{\phi}{2} \right]_{\phi_0}^{\phi_1} = 2\sqrt{gr} \sin^2 \frac{\psi_0}{2}.$$

The length of the line to be traversed from  $\phi_0$  to  $\phi_1$  is

$$x_1 - x_0 = r[\phi_1 - \phi_0 - \sin \phi_1 + \sin \phi_0] = 2r[\psi_0 - \sin \psi_0].$$

Hence, the time required is

$$T_1 = \frac{2r(\psi_0 - \sin \psi_0)}{2\sqrt{gr} \sin^2(\psi_0/2)},$$

and consequently

$$\frac{T_1}{T} = \frac{2r(\psi_0 - \sin \psi_0)}{2\sqrt{gr} \sin^2 \frac{\psi_0}{2} \cdot 2\sqrt{\frac{r}{g}} \psi_0} = \frac{\psi_0 - \sin \psi_0}{2\psi_0 \sin^2 \frac{\psi_0}{2}}$$

$$= \frac{\frac{\psi_0^3}{3!} - \frac{\psi_0^5}{5!} + \dots}{2\psi_0 \left[ \frac{\psi_0}{2} - \frac{1}{3!} \left( \frac{\psi_0}{2} \right)^3 + \dots \right]^2}.$$

Hence,

$$\frac{T_1}{T} < \frac{\frac{\psi_0^3}{3!}}{2\psi_0 \left[ \frac{\psi_0}{2} - \frac{1}{6} \left( \frac{\psi_0}{2} \right)^3 + \dots \right]^2}.$$

or

$$\frac{T_1}{T} < \frac{1}{3} \frac{1}{\left( 1 - \frac{\psi_0^2}{24} + \dots \right)^2}.$$

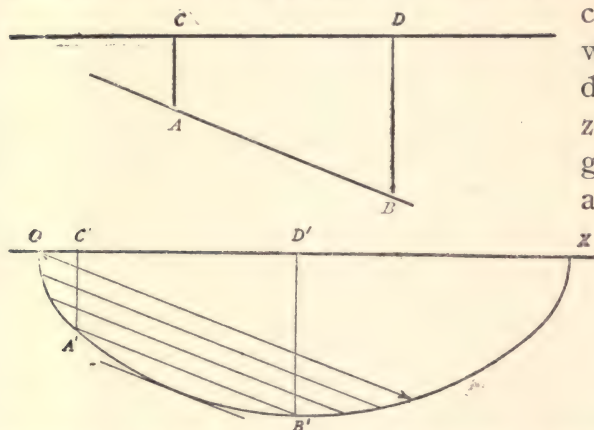
It follows, therefore, for small values of  $\psi_0$  that

$$T_1 < T.$$

From this it is evident that a path of the particle including an apex cannot give a minimum.

105. Corresponding to the two constants that are contained in the general solution of the differential equation  $G=0$  of Art. 103, it is seen that we have all the curves of the family  $G=0$ , if we vary  $r$  and slide the cycloid along the  $X$ -axis.

We shall now show that only one of these cycloids can contain the two points  $A$  and  $B$  on the same loop. Suppose that the ordinates of the points  $A$  and  $B$  to be such that  $DB > AC$ , and



consider any other cycloid with the same parameter  $r$  described about the horizontal  $X$ -axis with the origin at  $O$ . Through  $O$  draw a chord parallel to  $AB$  and move this chord through parallel positions until it leaves the curve. We note that in these positions the ordinate  $A'C'$  increases continuously,

since it can never reach the lowest point of the cycloid, and that the arc  $A'B'$  continuously diminishes. Consequently the ratio  $A'B':A'C'$  continuously diminishes. When  $A'$  coincides with the origin this ratio is infinite, and is zero when the chord becomes tangent to the curve.

Then for some one position we must have

$$\frac{A'B'}{A'C'} = \frac{AB}{AC}.$$

Since the points  $A$  and  $B$  are fixed, the length  $AB$  and the direction  $AB$  are both determined.

If  $A'B'=AB$ , then  $A'C'=AC$ , and a cycloid can be drawn through  $A$  and  $B$  as required. But if  $A'B' \neq AB$ , then our cycloid does not fulfill the required conditions.

Next choose a quantity  $r'$  such that

$$r:r'=AC:A'C'.$$

With  $O$  as the center of similitude increase the coordinates of our cycloid parameters in the ratio  $r:r'$ . These coordinates then become

$$\begin{aligned}x &= r'(\phi - \sin \phi), \\y &= r'(1 - \cos \phi),\end{aligned}$$

which are the coordinates of a new cycloid.

The latter cycloid is similar to the first, since the transformation moves the ordinate  $A'C'$  and the chord  $A'B'$  parallel to themselves. Their transformed lengths are respectively

$$\frac{r'}{r} A'C' = AC \quad \text{and} \quad \frac{r'}{r} A'B' = AB,$$

giving us a cycloid with the requisite lengths for the ordinate  $AC$  and the chord  $AB$ .

Further, there is but one cycloid which answers the required conditions. For, if we already had  $A'B' = AB$  and  $A'C' = AC$ , the only value of  $r'$  which could then make  $\frac{r'}{r} A'B' = AB$  is  $r' = r$ . Hence through the two points  $A$  and  $B$  there can be constructed one and only one cycloid-loop with respect to the  $X$ -axis.\*

106. PROBLEM III. *Problem of the shortest line on a surface.* This problem cannot in general be solved, since the variables in the differential equation cannot be separated and the integration cannot be performed. Only in a few instances has one succeeded in carrying out the integration and thus represented the curve which satisfies the differential equation.

This, for example, has been done in the case of the plane, the sphere and all the other surfaces of the second degree.

As a simple example, we will take the problem of the shortest line between two points on the surface of a sphere. The radius of the sphere is put equal to 1, and the equation of the sphere is given in the form

$$x^2 + y^2 + z^2 = 1.$$

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\* This proof is due to Prof. Schwarz.

Now writing:

$$\left. \begin{aligned} x &= \cos u, \\ y &= \sin u \cos v, \\ z &= \sin u \sin v, \end{aligned} \right\} \quad [1]$$

then  $u$ =constant and  $v$ =constant are the equations of the parallel circles and of the meridians respectively.

The element of arc is

$$ds = \sqrt{du^2 + \sin^2 u dv^2}, \quad [2]$$

and consequently the integral which is to be made a minimum is

$$L = \int_{t_0}^{t_1} \sqrt{u'^2 + v'^2 \sin^2 u} dt; \quad [3]$$

so that here we have

$$F = \sqrt{u'^2 + v'^2 \sin^2 u}, \quad [4]$$

and

$$\left. \begin{aligned} \frac{\partial F}{\partial u'} &= \frac{u'}{\sqrt{u'^2 + v'^2 \sin^2 u}}, \\ \frac{\partial F}{\partial v'} &= \frac{v' \sin^2 u}{\sqrt{u'^2 + v'^2 \sin^2 u}}. \end{aligned} \right\} \quad [5]$$

Since  $F$  does not contain the quantity  $v$ , we will use the equation  $G_1=0$ , and have:

$$\frac{\partial F}{\partial v'} = \frac{v' \sin^2 u}{\sqrt{u'^2 + v'^2 \sin^2 u}} = c,$$

where  $c$  is an arbitrary constant, which has the same value along the whole curve.

If for the initial point  $A$  of the curve  $u \neq 0$ , and consequently, therefore, *not* the north pole of the sphere, then  $c$  will be everywhere equal to zero, only if  $v'=0$ . We must therefore have  $v$  constant. It follows as a solution of the problem that  $A$  and  $B$  must lie on the same meridian.

If this is not the case, then always  $c \neq 0$ . It is easy to see that  $c < 1$ ; we may therefore write  $\sin c$  instead of  $c$ , and have

$$\frac{v' \sin^2 u}{\sqrt{u'^2 + \sin^2 u v'^2}} = \sin c, \quad [6]$$

or

$$dv = \frac{\sin c \, du}{\sin u \sqrt{\sin^2 u - \sin^2 c}}. \quad [7]$$

If we write

$$\cos u = \cos c \cos w, \quad [8]$$

then is

$$dv = \frac{\sin c \, dw}{1 - \cos^2 c \cos^2 w};$$

since 1 may be replaced by  $\sin^2 w + \cos^2 w$ , we have

$$dv = \frac{\sin c \, dw}{\sin^2 w + \cos^2 w \sin^2 c} = \frac{\sin c \frac{dw}{\cos^2 w}}{\sin^2 c + \tan^2 w} = \frac{d \frac{\tan w}{\sin c}}{1 + \frac{\tan^2 w}{\sin^2 c}}.$$

Therefore

$$v - \beta = \tan^{-1} \left( \frac{\tan w}{\sin c} \right),$$

where  $\beta$  represents an arbitrary constant.

It follows that

$$\tan (v - \beta) = \frac{\tan w}{\sin c}. \quad [9]$$

Eliminating  $w$  by means of [8], we have

$$\tan u \cos (v - \beta) = \tan c. \quad [10]$$

This is the equation of the curve which we are seeking, expressed in the spherical coordinates  $u, v$ .

In order to study their meaning more closely, we may express  $u, v$  separately through the arc  $s$ , where  $s$  is measured from the intersection of the zero meridian with the shortest line.

Through [7] the expression [2] goes into

$$ds = \frac{\sin u \, du}{\sqrt{\sin^2 u - \sin^2 c}},$$

and this, owing to the substitution [8], becomes

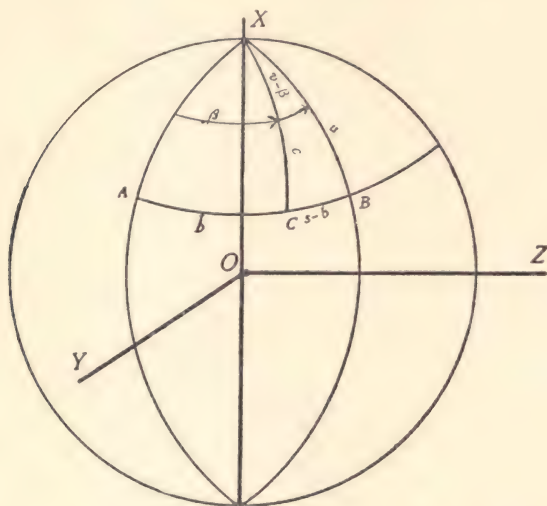
$$ds = dw,$$

and, therefore, if  $b$  is a new constant,

$$s - b = w. \quad [11]$$

Hence, from equations [8] and [9] we have the following equations :

$$\left. \begin{aligned} \cos u &= \cos c \cos (s - b), \\ \cot (v - \beta) &= \sin c \cot (s - b). \end{aligned} \right\} \quad [12]$$



But these are relations which exist among the sides and the angles of a right-angled spherical triangle.

If we consider the meridian drawn from the north pole, which cuts at right angles the curve we are seeking, then this meridian forms with the curve, and any other meridian, a triangle, to which the above relations may be applied.

Therefore, the curve which satisfies the differential equation must itself be the arc of a great circle. The constants of integration  $c$ ,  $b$ ,  $\beta$  are determined from the conditions that the curve is to pass through the two points  $A$  and  $B$ .

The geometrical interpretation is: that  $c$  is the length of the geodesic normal from the point  $u=0$  to the shortest line;  $s-b$ ,

the arc from the foot of this normal to any point of the curve, that is, the difference of length between the end-points of this arc; and  $v - \beta$ , the angle opposite this arc.

If we therefore assume that the zero meridian passes through  $A$ , then  $b$  is the length of arc of the shortest line from  $A$  to the normal, and  $\beta$  the geographical longitude of the foot of this normal.

107. We may derive the same results by considering the differential equation  $G = 0$ .

Since

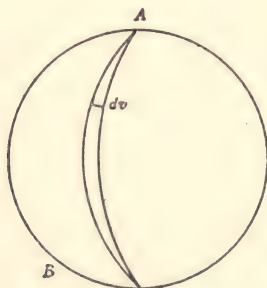
$$F_1 u'^2 = \frac{\partial^2 F}{\partial v'^2},$$

we have

$$F_1 = \frac{\sin^2 u}{(u'^2 + v'^2 \sin^2 u)^{3/2}}.$$

This value, substituted in

$$G \equiv \frac{\partial^2 F}{\partial v \partial u'} - \frac{\partial^2 F}{\partial u \partial v'} + F_1 (v' u'' - u' v'') = 0,$$



causes this expression to become

$$- [2 \cos u u'^2 v' + \sin^2 u \cos u v'^3] + \sin u (v' u'' - u' v'') = 0,$$

or

$$\frac{dv}{du} \left[ 2 + \sin^2 u \left( \frac{dv}{du} \right)^2 \right] \cos u + \sin u \frac{d^2 v}{du^2} = 0.$$

In this equation write

$$1) \quad w = \sin u \frac{dv}{du},$$

and we have

$$\cot u (w + w^3) + \frac{dw}{du} = 0,$$

or 2)

$$\frac{dw}{w + w^3} + \cot u \, du = 0.$$

Integrating the last equation, it follows that

$$\log \left( \frac{w}{\sqrt{1+w^2}} \sin u \right) = c,$$

and consequently,

$$3) \quad w^2 \sin^2 u = C^2 (1 + w^2).$$

Suppose that  $A$  is the north pole of the sphere,  $u$  the angular distance measured from  $A$  along the arc of a great circle, and  $v$  the angle which the plane of this great circle makes with the plane of a great circle through the point  $B$ .

Hence for all curves of the family  $G=0$  that pass through  $A$ , we must have  $C=0$ , since  $\sin u=0$  for  $u=0$ . It follows also that  $w=0$ , and consequently,

$$\sin u \frac{dv}{du} = 0,$$

or

$$v = \text{constant}.$$

Hence, as above,  $A$  and  $B$  must lie on the arc of a great circle. Next, if  $A$  is not taken as the pole, then always  $C \neq 0$ , and is less than unity. It follows then at once from equations 1) and 3) that

$$ds^2 = du^2 + \sin^2 u \, dv^2 = du^2 [1 + w^2],$$

or

$$ds = \pm \frac{\sin u \, du}{\sqrt{\sin^2 u - \sin^2 C}},$$

(where we have written  $\sin^2 C$  for  $C^2$ ),

and

$$dv = \frac{\sin C \, du}{\sin u \sqrt{\sin^2 u - \sin^2 C}}.$$

Writing  $\cos u = \cos C \cos t$ , these two equations when integrated become, as in the last article,

$$\begin{aligned}\cos u &= \cos C \cos (s-b), \\ \cot (v-\beta) &= \sin C \cot (s-b).\end{aligned}$$

108. PROBLEM IV. *Surface of rotation which offers the least resistance.* To solve this problem we saw (Art 12) that the integral

$$I = \int_{t_0}^{t_1} \frac{x x'^3}{x'^2 + y'^2} dt$$

must be a minimum.

We have here

$$F = \frac{x x'^3}{x'^2 + y'^2},$$

and we see that  $F$  is a rational function of the arguments  $x'$  and  $y'$ . For such functions Weierstrass has shown that there can never be a maximum or a minimum value of the integral. But leaving the general problem for a later discussion (Art. 173), we shall confine our attention to the problem before us.

We may determine the function  $F_1$  from the relation

$$\frac{\partial^2 F}{\partial x' \partial y'} = -x' y' F_1.$$

It is seen that

$$F_1 = \frac{2x x' (3y'^2 - x'^2)}{(x'^2 + y'^2)^3}.$$

We may take  $x$  positive, and also confine our attention to a portion of curve along which  $x$  increases with  $t$ , so that  $x'$  is also positive.

Consequently  $F_1$  has the same sign as  $3y'^2 - x'^2$ , or of  $3 \sin^2 \lambda - \cos^2 \lambda$ , where  $\lambda$  is the angle that the tangent to the curve at the point in question makes with the  $X$ -axis.

$F_1$  is therefore *positive*, if  $|\tan \lambda| > \frac{1}{\sqrt{3}}$ ,

and is *negative*, if  $|\tan \lambda| < \frac{1}{\sqrt{3}}$ ,

for the portion of curve considered.

We shall see later (Art. 117) that  $F_1$  must have a positive sign in order that the integral be a minimum. Hence, for the present problem,  $|\tan \lambda|$  must be greater than  $\frac{1}{\sqrt{3}}$  for the portion of curve considered; and as this must be true for all points of the curve at which  $x'$  has a positive sign, the tangent at any of these points cannot make an angle greater than  $30^\circ$  with the  $X$ -axis (see Todhunter, *Researches in the Calculus of Variations*, p. 168).

109. We shall next consider the differential equation  $G=0$  of the problem.

Since  $F$  does not contain explicitly the variable  $y$ , we may best employ the equation

$$-x'G = G_2 \equiv \frac{\partial F}{\partial y} - \frac{d}{dt} \frac{\partial F}{\partial y'} = 0.$$

We have at once

$$\frac{\partial F}{\partial y'} = \text{constant},$$

or

$$-\frac{2xx'^3y'}{(x'^2 + y'^2)^2} = C.$$

Now, if there is any portion of the surface offering resistance, which lies indefinitely near the axis of rotation, then the constant must be zero, since  $x=0$  makes  $C=0$ .

If  $C=0$ , we have

$$x'^3 y' = 0,$$

and consequently

$$x'=0 \text{ or } y'=0.$$

From this we derive

$$x=\text{const. or } y=\text{const.}$$

In the first case, the surface would be a cylinder of indefinite length, with the  $Y$ -axis as the axis of rotation, and with an indefinitely small radius (since by hypothesis a portion of the surface lies indefinitely near the  $Y$ -axis); in the second case, the resisting-surface would be a disc of indefinitely large diameter. These solutions being without significance may be neglected, and we

may, therefore, suppose that the surface offering resistance has no points in the neighborhood of the  $Y$ -axis. This disproves the notion once held that the body was egg-shaped.

110. We consider next the differential equation

$$-\frac{2x x'^3 y'}{(x'^2 + y'^2)^2} = C,$$

where  $C$  is different from zero. We may take  $x$  positive, and as the constant  $C$  must always retain the same sign (Art. 97), it follows that the product  $x'y'$  cannot change sign.

Instead of retaining the variable  $t$ , let us write

$$t = -y,$$

and

$$\frac{dx}{dy} = u.$$

The differential equation is then

$$\frac{-2x u^3}{(u^2 + 1)^2} = C.$$

That we may write  $-y$  in the place of  $t$ , is seen from the fact that  $x'y'$  cannot change sign, and consequently either  $x$  is *continuously* increasing with increasing  $y$ , or is *continuously* decreasing when  $y$  is increased. Hence, corresponding to a given value of  $y$  there is one value of  $x$ .

We have then

$$x = -\frac{C(u^2 + 1)^2}{2u^3} = -\frac{C}{2}(u + 2u^{-1} + u^{-3}),$$

and

$$\frac{dx}{du} = \frac{dx}{dy} \frac{dy}{du} = u \frac{dy}{du} = -\frac{C}{2}(1 - 2u^{-2} - 3u^{-4}),$$

or

$$\frac{dy}{du} = -\frac{C}{2}(u^{-1} - 2u^{-3} - 3u^{-5});$$

consequently

$$y = -\frac{C}{2} \left[ \log u + u^{-2} + \frac{3}{4} u^{-4} \right] + C_1.$$

The equations

$$\begin{cases} x = -\frac{C}{2} (u + 2u^{-1} + u^{-3}), \\ y = -\frac{C}{2} (\log u + u^{-2} + \frac{3}{4}u^{-4}) + C_1, \end{cases}$$

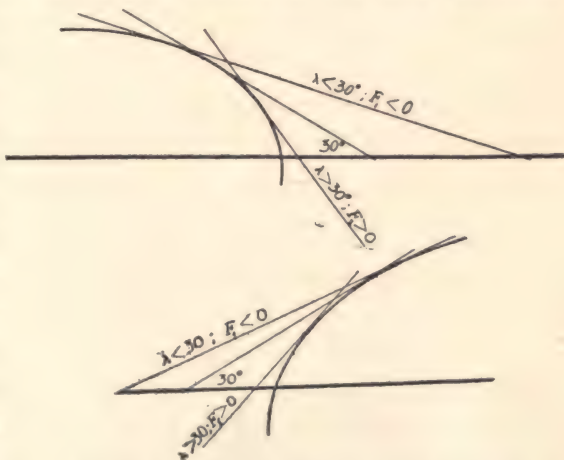
determine a family of curves, one of which is the arc, which generates the surface of revolution that gives a minimum value, if such a minimum exists. For such a curve we have all the real points if we give to  $u$  all real values from 0 to  $+\infty$ . Among these values is  $\sqrt{3}$ , and as we saw above, it is necessary that

$$\frac{dy}{dx} = \frac{1}{u} > \frac{1}{\sqrt{3}} \text{ continuously,}$$

or

$$\frac{1}{u} < \frac{1}{\sqrt{3}} \text{ continuously.}$$

In other words, if the acute angle which the tangent at any point of the arc makes with the  $X$ -axis is less than  $30^\circ$ , it must continue less than  $30^\circ$  for the other points of the arc, and if it is greater than  $30^\circ$  for any point of the arc, it must remain greater than  $30^\circ$  for all points of the arc. Hence, if  $P$  is the point at which the inclination of the tangent with the  $X$ -axis is  $30^\circ$ , we shall have on one side of  $P$  that portion of curve for which the inclination is less than  $30^\circ$ , and on the other side the portion of curve for which the inclination is greater than  $30^\circ$ . The arc in question must belong entirely to one of the two portions.



## CHAPTER VIII.

THE SECOND VARIATION; ITS SIGN DETERMINED BY THAT  
OF THE FUNCTION  $F_1$ .

111. The substitution  $x + \epsilon \xi, y + \epsilon \eta$  for  $x, y$  causes any point of the original curve to move along a straight line, which makes an angle with the  $X$ -axis whose tangent is  $\frac{\eta}{\xi}$ .

This deformation of the curve is insufficient, if we require that the point move along a curve other than a straight line.

To avoid this inadequacy we make the more general substitution (by which the regular curve remains regular):

$$\begin{array}{l} x \parallel x + \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots, \\ y \parallel y + \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots, \end{array}$$

where, like  $\xi, \eta$  in our previous development (Art. 75), the quantities  $\xi_1, \eta_1, \xi_2, \eta_2, \dots$  are functions of  $t$ , finite, continuous, one-valued and capable of being differentiated (as far as necessary) between the limits  $t_0 \dots t_1$ . These series are supposed to be convergent for values of  $\epsilon$  such that  $|\epsilon| < 1$ .

That such substitutions exist may be seen as follows:

Since the curve is regular, the coordinates of consecutive points to  $P_0$  and  $P_1$  may be expressed by series in the form, say,

$$(A) \left\{ \begin{aligned} x_0 + \epsilon a_0^{(1)} + \frac{\epsilon^2}{2!} a_0^{(2)} + \dots, \\ y_0 + \epsilon b_0^{(1)} + \frac{\epsilon^2}{2!} b_0^{(2)} + \dots, \end{aligned} \right.$$

$$(B) \left\{ \begin{aligned} x_1 + \epsilon a_1^{(1)} + \frac{\epsilon^2}{2!} a_1^{(2)} + \dots, \\ y_1 + \epsilon b_1^{(1)} + \frac{\epsilon^2}{2!} b_1^{(2)} + \dots, \end{aligned} \right.$$

where the coefficients of the powers of  $\epsilon$  are constants and the series are convergent.

Suppose, now, that we seek to determine the functions of  $t$

$$(C) \left\{ \begin{aligned} x + \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots, \\ y + \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots, \end{aligned} \right.$$

such that for  $t=t_0$  and  $t=t_1$ , the expressions (C) will be the same as (A) and (B).

This may be done, for example, by writing

$$\xi_1 = t^2 + \alpha_1 t + \alpha_2,$$

$$\eta_1 = t^2 + \beta_1 t + \beta_2,$$

and then determine  $\alpha_1, \alpha_2, \beta_1, \beta_2$  in such a way that

$$t_0^2 + \alpha_1 t_0 + \alpha_2 = a_0^{(1)}; \quad t_0^2 + \beta_1 t_0 + \beta_2 = b_0^{(1)},$$

$$t_1^2 + \alpha_1 t_1 + \alpha_2 = a_1^{(1)}; \quad t_1^2 + \beta_1 t_1 + \beta_2 = b_1^{(1)}.$$

From this it is seen that

$$\alpha_1 = -(t_1 + t_0) + \frac{a_1^{(1)} - a_0^{(1)}}{t_1 - t_0}, \text{ etc.}$$

In the same way we may determine quadratic expressions in  $t$  for  $\xi_2, \eta_2$ , etc.

The substitutions thus obtained are of the nature of those which we have assumed to exist, and may evidently be constructed in an infinite number of different ways.

112. Making the above substitutions in the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt,$$

it is seen that

$$\begin{aligned} \Delta I &= \int_{t_0}^{t_1} \left[ F \left( x + \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots, y + \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots, \right. \right. \\ &\quad \left. \left. x' + \epsilon \xi_1' + \frac{\epsilon^2}{2!} \xi_2' + \dots, y' + \epsilon \eta_1' + \frac{\epsilon^2}{2!} \eta_2' + \dots \right) \right. \\ &\quad \left. - F(x, y, x', y') \right] dt \\ &= \epsilon \delta I + \frac{\epsilon^2}{2!} \delta^2 I + \frac{\epsilon^3}{3!} \delta^3 I + \dots \end{aligned}$$

By Taylor's Theorem we have

$$\begin{aligned} &F \left( x + \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots, y + \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots, x' + \epsilon \xi_1' + \frac{\epsilon^2}{2!} \xi_2' + \dots, \right. \\ &\quad \left. y' + \epsilon \eta_1' + \frac{\epsilon^2}{2!} \eta_2' + \dots \right) - F(x, y, x', y') \\ &= \left[ \left( \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots \right) \frac{\partial}{\partial x} + \left( \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots \right) \frac{\partial}{\partial y} \right. \\ &\quad \left. + \left( \epsilon \xi_1' + \frac{\epsilon^2}{2!} \xi_2' + \dots \right) \frac{\partial}{\partial x'} + \left( \epsilon \eta_1' + \frac{\epsilon^2}{2!} \eta_2' + \dots \right) \frac{\partial}{\partial y'} \right] F \\ &+ \frac{1}{2!} \left[ \left( \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots \right) \frac{\partial}{\partial x} + \left( \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots \right) \frac{\partial}{\partial y} \right. \\ &\quad \left. + \left( \epsilon \xi_1' + \frac{\epsilon^2}{2!} \xi_2' + \dots \right) \frac{\partial}{\partial x'} + \left( \epsilon \eta_1' + \frac{\epsilon^2}{2!} \eta_2' + \dots \right) \frac{\partial}{\partial y'} \right]^2 F \\ &+ \frac{1}{3!} \left[ \dots \right]^3 F + \dots \end{aligned}$$

The coefficient of  $\epsilon$  in this expression is the integrand of  $\delta I$  and is zero; while the coefficient of  $\frac{\epsilon^2}{2!}$  involves terms that are the first partial derivatives of  $F$ , and also those that are the second partial derivatives of  $F$ .

The first partial derivatives of  $F$  that belong to this coefficient, when put under the integral sign, may be written in the form

$$\begin{aligned} & \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial x} \xi_2 + \frac{\partial F}{\partial x'} \xi_2' + \frac{\partial F}{\partial y} \eta_2 + \frac{\partial F}{\partial y'} \eta_2' \right] dt \\ &= \int_{t_0}^{t_1} G (y' \xi_2 - x' \eta_2) dt + \left[ \frac{\partial F}{\partial x'} \xi_2 + \frac{\partial F}{\partial y'} \eta_2 \right]_{t_0}^{t_1} \end{aligned}$$

(see Art. 79), and this expression is also zero, if we suppose that the end-points remain fixed.

113. The coefficient of  $\epsilon^2$  in the preceding development of  $F$  by Taylor's Theorem is, neglecting the factor  $\frac{1}{2!}$ , denoted by  $\delta^2 F$ .

We have then

$$\begin{aligned} 1) \quad \delta^2 F &= \frac{\partial^2 F}{\partial x^2} \xi_1^2 + 2 \frac{\partial^2 F}{\partial x \partial y} \xi_1 \eta_1 + \frac{\partial^2 F}{\partial y^2} \eta_1^2 + \frac{\partial^2 F}{\partial x'^2} \xi_1'^2 + 2 \frac{\partial^2 F}{\partial x' \partial y'} \xi_1' \eta_1' \\ &+ \frac{\partial^2 F}{\partial y'^2} \eta_1'^2 + 2 \left( \frac{\partial^2 F}{\partial x \partial x'} \xi_1 \xi_1' + \frac{\partial^2 F}{\partial y \partial y'} \eta_1 \eta_1' + \frac{\partial^2 F}{\partial x \partial y'} \xi_1 \eta_1' \right. \\ &\left. + \frac{\partial^2 F}{\partial y \partial x'} \eta_1 \xi_1' \right). \end{aligned}$$

The subscripts may now be omitted and the formula simplified by the introduction of the function  $F_1$ , which (Art. 73) was defined by the relations:

$$2) \quad \begin{cases} \frac{\partial^2 F}{\partial x'^2} = y'^2 F_1, \\ \frac{\partial^2 F}{\partial x' \partial y'} = -x' y' F_1, \\ \frac{\partial^2 F}{\partial y'^2} = x'^2 F_1; \end{cases}$$

and by introducing the new notation:

$$3) \begin{cases} L = \frac{\partial^2 F}{\partial x \partial x'} - y' y'' F_1, \\ M = \frac{\partial^2 F}{\partial x \partial y'} + x' y'' F_1 = \frac{\partial^2 F}{\partial x' \partial y} + y' x'' F_1 \quad (\text{owing to the equation } G=0), \\ N = \frac{\partial^2 F}{\partial y \partial y'} - x' x'' F_1; \end{cases}$$

where  $x'', y''$  are used for  $\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}$ .

We have then

$$\begin{aligned} \delta^2 F = & \frac{\partial^2 F}{\partial x^2} \xi^2 + 2 \frac{\partial^2 F}{\partial x \partial y} \xi \eta + \frac{\partial^2 F}{\partial y^2} \eta^2 + F_1 (y'^2 \xi'^2 - 2x' y' \xi' \eta' + x'^2 \eta'^2) \\ & + 2 F_1 (y' y'' \xi \xi' + x' x'' \eta \eta' - x' y'' \xi \eta' - y' x'' \eta \xi') \\ & + 2 \{ L \xi \xi' + M (\xi \eta' + \eta \xi') + N \eta \eta' \}. \end{aligned}$$

To get an exact differential as a part of the right-hand member of this formula, we write

$$4) \quad R = L \xi^2 + 2M \xi \eta + N \eta^2,$$

an expression which, differentiated with respect to  $t$ , becomes

$$2[L \xi \xi' + M(\xi \eta' + \eta \xi') + N \eta \eta'] = \frac{dR}{dt} - \frac{dL}{dt} \xi^2 - \frac{2dM}{dt} \xi \eta - \frac{dN}{dt} \eta^2.$$

We further write

$$5) \quad w = y' \xi - \eta x',$$

where (see Art. 81)  $w$  is, neglecting the factor  $\frac{1}{\sqrt{x'^2 + y'^2}}$ , the amount of the sliding of a point of the curve in the direction of the normal.

Differentiating with respect to  $t$ , we have

$$\frac{dw}{dt} = y'' \xi - x'' \eta + y' \xi' - x' \eta',$$

from which it follows that

$$\begin{aligned} \left(\frac{dw}{dt}\right)^2 &= (y'' \xi - x'' \eta)^2 + (y' \xi' - x' \eta')^2 \\ &\quad + 2(y' y'' \xi \xi' + x' x'' \eta \eta' - x' y'' \xi \eta' - y' x'' \eta \xi'). \end{aligned}$$

Then the expression for the second variation becomes

$$\begin{aligned} \delta^2 F &= \frac{\partial^2 F}{\partial x^2} \xi^2 + 2 \frac{\partial^2 F}{\partial x \partial y} \xi \eta + \frac{\partial^2 F}{\partial y^2} \eta^2 + F_1 \left\{ \left(\frac{dw}{dt}\right)^2 - (y'' \xi - x'' \eta)^2 \right\} \\ &\quad + \frac{dR}{dt} - \left( \frac{dL}{dt} \xi^2 + 2 \frac{dM}{dt} \xi \eta + \frac{dN}{dt} \eta^2 \right). \end{aligned}$$

If further we write in this expression

$$6) \quad \begin{cases} L_1 = \frac{\partial^2 F}{\partial x^2} - F_1 y''^2 - \frac{dL}{dt}, \\ M_1 = \frac{\partial^2 F}{\partial x \partial y} + F_1 x'' y'' - \frac{dM}{dt}, \\ N_1 = \frac{\partial^2 F}{\partial y^2} - F_1 x''^2 - \frac{dN}{dt}, \end{cases}$$

we have finally

$$\delta^2 F = F_1 \left(\frac{dw}{dt}\right)^2 + L_1 \xi^2 + 2 M_1 \xi \eta + N_1 \eta^2 + \frac{dR}{dt}.$$

114. It follows from 3) that

$$L x' + M y' = x' \frac{\partial^2 F}{\partial x \partial x'} + y' \frac{\partial^2 F}{\partial x \partial y'}.$$

Owing to the homogeneity of the function  $F$  (Chap. IV), it is seen from Euler's Theorem that

$$F = x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'},$$

and consequently,

$$\frac{\partial F}{\partial x} = x' \frac{\partial^2 F}{\partial x \partial x'} + y' \frac{\partial^2 F}{\partial x \partial y'};$$

and therefore

$$\frac{\partial F}{\partial x} = Lx' + My'.$$

In a similar manner we have

$$\frac{\partial F}{\partial y} = Mx' + Ny'.$$

Differentiating with regard to  $t$ , the above expression becomes

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial F}{\partial x} \right) &= \frac{\partial^2 F}{\partial x^2} x' + \frac{\partial^2 F}{\partial x \partial y} y' + \frac{\partial^2 F}{\partial x \partial x'} x'' + \frac{\partial^2 F}{\partial x \partial y'} y'' \\ &= \frac{dL}{dt} x' + \frac{dM}{dt} y' + L x'' + M y'', \end{aligned}$$

which, owing to 3), is

$$x' \left\{ \frac{\partial^2 F}{\partial x^2} - F_1 y'^2 - \frac{dL}{dt} \right\} + y' \left\{ \frac{\partial^2 F}{\partial x \partial y} + F_1 y'' x'' - \frac{dM}{dt} \right\} = 0;$$

or from 6)

$$x' L_1 + y' M_1 = 0.$$

In an analagous manner it may be shown that

$$x' M_1 + y' N_1 = 0.$$

From these expressions we have at once

$$\frac{L_1}{y'^2} = -\frac{M_1}{x' y'} = \frac{N_1}{x'^2} = F_2,$$

where  $F_2$  is the factor of proportionality.

It follows that

$$7) \quad \begin{cases} L_1 = y'^2 F_2, \\ M_1 = -x' y' F_2, \\ N_1 = x'^2 F_2. \end{cases}$$

The quantity  $F_2$  is defined through these three equations and plays an essential role in the treatment of the second variation.

Owing to the relation 7)

$$L_1 \xi^2 + 2 M_1 \xi \eta + N_1 \eta^2 \text{ becomes } F_2 w^2,$$

and consequently,

$$\delta^2 F = F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 + \frac{dR}{dt}.$$

115. The second variation of the integral has therefore the form

$$8) \quad \delta^2 I = \int_{t_0}^{t_1} \left\{ F \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right\} dt + \int_{t_0}^{t_1} \frac{dR}{dt} dt.$$

We suppose that the end-points are fixed so that at these points  $\xi = 0 = \eta$ , and we further assume that the curve subjected to variation consists of a single regular trace, along which then

$$R = L \xi^2 + 2 M \xi \eta + N \eta^2$$

is everywhere continuous, so that

$$\left[ R \right]_{t_0}^{t_1} = 0.$$

Consequently the above integral may be written

$$8^a) \quad \delta^2 I = \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right\} dt.$$

If the integral  $I = \int_{t_0}^{t_1} F(x, y, x', y') dt$  is to be a maximum or

a minimum for the curve  $G=0$ , it is necessary, when the curve is subjected to an indefinitely small variation, that the variation  $\Delta I$ , which is caused to exist therefrom, have always the same sign, in whatever manner  $\xi, \eta$  are chosen; and consequently the second variation  $\delta^2 I$  must have continuously the same sign as  $\Delta I$ .

We have repeatedly seen that

$$\Delta I = \frac{\epsilon^2}{2!} \delta^2 I + \frac{\epsilon^3}{3!} \delta^3 I + \dots,$$

and for any other value of  $\epsilon$ , for example,  $\epsilon_1$ ,

$$\Delta_1 I = \frac{\epsilon_1^2}{2!} \delta^2 I + \frac{\epsilon_1^3}{3!} \delta^3 I + \dots$$

If, further,  $\delta^2 I$  is negative while  $\Delta I$  is positive, then we may take  $\epsilon_1$  so small that the sign of  $\Delta_1 I$  depends only upon the first term on the right in the above expansion, and consequently is negative. Therefore the integral  $I$  cannot be a maximum or a minimum, since the variation of it is first positive and then negative.

Hence, neglecting for a moment the case when  $\delta^2 I = 0$ , we have the following theorem:

*If the integral  $I$  is to be a maximum or a minimum, its second variation must be continuously negative or continuously positive.*

When  $\delta^2 I$  vanishes for all possible values of  $\xi, \eta$ , it is necessary also that  $\delta^3 I$  vanish, since the integral  $I$  is to be a maximum or a minimum, and, as in the Theory of Maxima and Minima, we would then have to investigate the fourth variation. In this case the conditions that have to be satisfied are so numerous that a mathematical treatment is very complicated and difficult.

Hence, it is seen that after the condition  $\delta I = 0$  is satisfied, it follows that

for the possibility of a maximum,  $\delta^2 I$  must be negative, and  
for the possibility of a minimum,  $\delta^2 I$  must be positive.

These conditions are necessary, but not sufficient.

116. In Art. 75 we assumed that  $\xi, \eta, \xi', \eta'$  were continuous functions of  $t$  between the limits  $t_0 \dots t_1$ . Owing to the assumed existence of  $\xi', \eta'$ , we must presuppose the existence of the second derivatives of  $x$  and  $y$  with respect to  $t$  (see Art. 23). From this

it also follows that the radius of curvature must vary in a continuous manner. These assumptions have been tacitly made in the derivation of the equation 8) in the preceding article. We shall now free ourselves from the restriction that  $\xi'$  and  $\eta'$  are continuous functions of  $t$ , retaining, however, the assumptions regarding the continuity of the quantities  $x, y, \xi, \eta, x', y', x'', y''$ .

The theorem that  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  vary in a continuous manner for the whole curve (Art. 97) in most cases gives a handy means of determining the admissibility of assumptions regarding the continuity of  $x'$  and  $y'$ . If, at certain points of the curve  $G=0$ ,  $x'$  and  $y'$  are not continuous, it is always possible to divide the curve into such portions that  $x'$  and  $y'$  are continuous throughout each portion. Yet we cannot even then say that  $x''$  and  $y''$  are continuous within such a portion, as has been assumed to be true in the above development. If, however,  $x''$  and  $y''$  within such a portion of curve are discontinuous, we have only to divide the curve into other portions so that within these new portions  $x''$  and  $y''$  no longer suffer any sudden springs. In each of these portions of curve the same conclusions may be made as before in the case of the whole curve, and consequently the assumption regarding the continuous change of  $x'', y''$  throughout the whole curve is not necessary. But if we had limited ourselves to the consideration of a part of the curve in which  $x, y, x', y', x'', y''$  vary in a continuous manner, the continuity of  $\xi', \eta'$  in the integration of the integral

$$\int \frac{dR}{dt} dt$$

would have been assumed. These assumptions need not necessarily be fulfilled, since the variation of the curve is an arbitrary one, and it is quite possible that such variations may be introduced, where  $\xi', \eta'$  become discontinuous, as often as we please. We may, however, drop these assumptions without changing the final results, if only the first named conditions are satisfied. Since the quantities  $L, M, N$  depend only upon  $x, y, x', y', x'', y''$ , and since these quantities are continuous, it follows that the introduction of the integral  $\int \frac{dR}{dt} dt$  in the form given above is always

admissible. For if  $\xi', \eta'$  were not continuous for the whole trace of the curve, which has been subjected to variation, we could suppose that this curve has been divided into parts, within which the above derivatives varied in a continuous manner, and the integral would then become a sum of integrals of the form

$$\int_{t_\beta}^{t_{\beta+1}} \frac{dR}{dt} dt = \left[ L \xi^2 + 2 M \xi \eta + N \eta^2 \right]_{t_\beta}^{t_{\beta+1}},$$

where  $t_\beta, t_{\beta+1}, \dots$  are the coordinates of the points of division of corresponding values of  $t$ . But since  $\xi, \eta$  vary in a continuous manner, we have through the summation of these quantities exactly the same expression

$$\left[ L \xi^2 + 2 M \xi \eta + N \eta^2 \right]_{t_0}^{t_1}$$

as before. The quantities  $\xi', \eta'$  are also found under the sign of integration in the right-hand side of 8); but owing to the conception of a definite integral, we may still write it in this form, even when these quantities vary in a discontinuous manner; however, in performing the integration, we must divide the integral corresponding to the positions at which the discontinuities enter into partial integrals. Therefore, we see that the possible discontinuity of  $\xi', \eta'$  remains without influence upon the result, if only  $x, y, x' y', x'', y'', \xi, \eta$  are continuous. Consequently any assumptions regarding the continuity of  $\xi', \eta'$  are superfluous; however, in an arbitrarily small portion of the curve which is subjected to variation, the quantities  $\xi'$  and  $\eta'$  must not become discontinuous an infinite number of times, since such variation of the curve has been, necessarily, once for all excluded.

117. Following the older mathematicians, Legendre, Jacobi, etc., we may give the second variation a form in which all terms appearing under the sign of integration will have the same sign (plus or minus).

To accomplish this, we add an exact differential  $\frac{d}{dt}(u^2 v)$

under the integral sign in 8), and subtract it from  $R$ , the integral thus becoming

$$\delta^2 I = \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + 2vw \frac{dw}{dt} + \left( F_2 + \frac{dv}{dt} \right) w^2 \right\} dt + \left[ R - vw^2 \right]_{t_0}^{t_1}.$$

The expression under the sign of integration is an integral homogeneous quadratic form in  $w$  and  $\frac{dw}{dt}$ . We choose the quantity  $v$  so that this expression becomes a perfect square; that is,

$$9) \quad v^2 - F_1 \left( F_2 + \frac{dv}{dt} \right) = 0,$$

and consequently,

$$10) \quad \delta^2 I = \int_{t_0}^{t_1} F_1 \left( \frac{dw}{dt} + w \frac{v}{F_1} \right)^2 dt + \left[ R - vw^2 \right]_{t_0}^{t_1}.$$

We shall see that it is possible to determine a function  $v$ , which is finite, one-valued and continuous within the interval  $t_0 \dots t_1$ , and which satisfies the equation 9). The integral 10) becomes accordingly, if the end-points remain fixed,

$$10^a) \quad \delta^2 I = \int_{t_0}^{t_1} F_1 \left( \frac{dw}{dt} + w \frac{v}{F_1} \right)^2 dt.$$

Hence the second variation has the same sign as  $F_1$ , and it is clear that *for the existence of a maximum  $F_1$  must be negative, and for a minimum this function must be positive within the interval  $t_0 \dots t_1$ , and in case there is a maximum or a minimum,  $F_1$  cannot change sign within this interval.*

This condition is due to Jacobi. Legendre had previously concluded that we have a maximum when a certain expression corresponding to  $F_1$  was negative, and a minimum when it was posi-

tive. It is questionable whether the differential equation for  $v$  is always integrable. Following Jacobi we shall show that such is the case.

118. Before we go farther, we have yet to prove that the transformation, which we have introduced, is allowable. In spite of the simplicity of the equation 9) we cannot make conclusions regarding the continuity of the function  $v$ , which is necessary for the above transformation. It is therefore essential to show that the equation 9) may be reduced to a system of two linear differential equations, which may be reverted into a linear differential equation of the second order, since for this equation we have definite criteria of determining whether a function which satisfies it remains finite and continuous or not.

Write

$$v = \frac{u_1}{u},$$

where  $u_1$  and  $u$  are continuous functions of  $t$ , and  $u \neq 0$  within the interval  $t_0 \dots t_1$ .

Equation 9) becomes then

$$\frac{u_1^2}{u^2} - F_1 \left\{ F_2 + \frac{u \frac{du_1}{dt} - u_1 \frac{du}{dt}}{u^2} \right\} = 0,$$

or

$$F_1 u \left\{ \frac{du_1}{dt} + F_2 u \right\} - u_1 \left\{ F_1 \frac{du}{dt} + u_1 \right\} = 0.$$

Since one of the functions  $u, u_1$  may be arbitrarily chosen, we take  $u$  so that

$$11) \quad F_1 \frac{du}{dt} + u_1 = 0;$$

then, since  $u \neq 0$ , we have

$$12) \quad \frac{du_1}{dt} + F_2 u = 0.$$

From 11) and 12) it follows that

$$12^a) \quad \frac{d}{dt} \left( F_1 \frac{du}{dt} \right) - F_2 u = 0,$$

or

$$13) \quad F_1 \frac{d^2 u}{dt^2} + \frac{dF_1}{dt} \frac{du}{dt} - F_2 u = 0,$$

where  $F_1$  and  $F_2$  are to be considered as given functions of  $t$ . We shall denote this differential equation by  $J=0$ . After  $u$  has been determined from this equation,  $u_1$  may be determined from 11), and from  $\frac{u_1}{u} = v$  we have  $v$  as a definite function of  $t$ .

119. The expression which has been derived for  $v$  seems to contain two arbitrary constants, while the equation 9) has only one. The two constants in the first case, however, may be replaced by one, since the general solution of 13) is

$$u = c_1 \phi_1(t) + c_2 \phi_2(t),$$

and hence from 11)

$$v = \frac{u_1}{u} = -F_1 \frac{c_1 \phi_1'(t) + c_2 \phi_2'(t)}{c_1 \phi_1(t) + c_2 \phi_2(t)},$$

an expression which depends only upon the ratio of the two constants.

It follows from the above transformation that

$$14) \quad \delta^2 I = \int_{t_0}^{t_1} F_1 \left( \frac{dw}{dt} - \frac{w}{u} \frac{du}{dt} \right)^2 dt;$$

but this transformation has a meaning only when it is possible to find a function  $u$  within the interval  $t_0 \dots t_1$ , which is different from zero, and which satisfies the differential equation  $J=0$ .

120. If we have a linear differential equation of the second order

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0,$$

and if  $y_1$  and  $y_2$  are a fundamental system of integrals of this equation, then we have the well known relation due to Abel (see Forsyth's Differential Equations, p. 99)

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = Ce^{-\int P(x) dx},$$

or

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = Ce^{-\int P(x) dx}.$$

If  $\Delta = 0$ , then we would have  $y_1 = cy_2$ , and the system is no longer a fundamental system of integrals. This determinant can become zero only at such positions for which  $P(x)$  becomes infinitely large; or a change of sign for this determinant can enter only at such positions where  $P(x)$  becomes infinite.

In the differential equation  $J = 0$  we have  $P = \frac{d}{dt} (\log F_1)$ , and if  $u_1, u_2$  form a fundamental system of integrals of this differential equation, then

$$\Delta = u_1 \frac{du_2}{dt} - u_2 \frac{du_1}{dt} = \frac{C}{F_1}.$$

It follows that  $F_1$  cannot become infinite or zero within the interval under consideration or upon the boundaries of this interval. Hence, it is again seen that  $F_1$  cannot change sign within the interval  $t_0 \dots t_1$ .

If  $F_1$  and  $F_2$  are continuous within the interval  $t_0 \dots t_1$ , we have, through differentiating the equation  $J = 0$ , all higher derivatives of  $u$  expressed in terms of  $u$  and  $\frac{du}{dt}$ . Hence, if values of  $u$  and  $\frac{du}{dt}$  are given for a definite value of  $t$ , say  $t'$ , we have a power-series  $P(t - t')$  for  $u$  (see Art. 79), which satisfies the equation  $J = 0$ .

121. Suppose that  $F_1$  has a definite, positive or negative value for a definite value  $t'$  of  $t$  situated within the interval  $t_0 \dots t_1$ , then on account of its continuity it will also be positive

or negative for a certain neighborhood of  $t'$ , say  $t' - \tau_1 \dots t' + \tau_2$ . We may vary the curve in such a manner that within the interval  $t' - \tau_1 \dots t' + \tau_2$  it takes any form, while without this region it remains unchanged.

Consequently the total variation, and therefore also the second variation of  $I$ , depends only upon the variation within the region just mentioned, and in accordance with the remarks made above, since we may find a function  $u$  of the variable  $t$ , which is continuous within the given region, which satisfies the differential equation  $J=0$ , and which is of such a nature that  $u$  and  $\frac{du}{dt}$  have given values for  $t=t'$ , it follows that the transformation which was introduced is admissible, and we have

$$\delta^2 I = \int_{t_0}^{t_1} F_1 \left\{ \frac{dw}{dt} - \frac{du}{dt} \frac{w}{u} \right\}^2 dt.$$

This quantity is evidently positive when  $F_1$  is positive, and negative when  $F_1$  is negative, so long as

$$\left\{ \frac{dw}{dt} - \frac{du}{dt} \frac{w}{u} \right\} \neq 0 \text{ (Art. 132).}$$

We have then for the total variation

$$\Delta I = \frac{\epsilon^2}{2!} \int_{t_0}^{t_1} F_1 \left\{ \frac{dw}{dt} - \frac{du}{dt} \frac{w}{u} \right\}^2 dt + \frac{\epsilon^3}{3!} \int_{t_0}^{t_1} (\xi, \eta, \xi', \eta')_3 dt,$$

where  $(\xi, \eta, \xi', \eta')_3$  denotes an expression of the third dimension in the quantities included within the brackets.

For small values of  $\epsilon$  it is seen that  $\Delta I$  has the same sign as the first term on the right-hand side of the above equation. We have, therefore, the following theorem:

*The total variation  $\Delta I$  of the integral  $I$  is positive when  $F_1$  is positive, and negative when  $F_1$  is negative throughout the whole interval  $t_0 \dots t_1$ .*

If  $F_1$  could change sign for any position within the interval  $t_0 \dots t_1$ , then there would be variations of the curve for which  $\Delta I$  is positive, and others for which  $\Delta I$  is negative. Hence, for the existence of a maximum or a minimum of  $I$  we have the following necessary condition:

*In order that there exist a maximum or a minimum of the integral  $I$  taken over the curve  $G=0$  within the interval  $t_0 \dots t_1$ , it is necessary that  $F_1$  have always the same sign within this interval; in the case of a maximum  $F_1$  must be continuously negative, and in the case of a minimum this function must be continuously positive.*

In this connection it is interesting to note a paper by Prof. W. F. Osgood in the Transactions of the American Mathematical Society, Vol. II, p. 273, entitled:

*"On a fundamental property of a minimum in the Calculus of Variations and the proof of a theorem of Weierstrass's."*

This paper, which is of great importance, may be much simplified.

## CHAPTER IX.

### CONJUGATE POINTS.

122. The condition given in the preceding Chapter is not sufficient to establish the existence of a maximum or a minimum. Under the assumption that  $F_1$  is neither zero nor infinite within the interval  $t_0 \dots t_1$ , suppose that two functions  $\phi_1(t)$  and  $\phi_2(t)$  can be found which satisfy the differential equation 13) of the last Chapter, so that, consequently,

$$u = c_1 \phi_1(t) + c_2 \phi_2(t)$$

is the general solution of  $J = 0$ . Then, even if within the limits of integration it can be shown that  $u$  is not infinite, it may still happen that, however the constants  $c_1$  and  $c_2$  be chosen, the function  $u$  vanishes, so that the transformation of the  $v$ -equation into the  $u$ -equation is not admissible; consequently nothing can be determined regarding the appearance of a maximum or a minimum. We are thus led again to the necessity of studying more closely the function  $u$  defined by the equation  $J = 0$ , in order that we may determine under what conditions this function does not vanish within the interval  $t_0 \dots t_1$ .

It is seen that the equation  $J = 0$  is satisfied, if for  $u$  we write

$$u_1 = -F_1 u' \text{ [see Art. 118, equation 11)],}$$

and consequently

$$v = \frac{u_1}{u} = -F_1 \frac{u'}{u}$$

is a solution of the equation in  $v$ .

The integral 10) of the last Chapter may be then written

$$\delta^2 I = \int_{t_0}^{t_1} F_1 w^2 \left( \frac{w'}{w} - \frac{u'}{u} \right)^2 dt + \left[ R + w^2 F_1 \frac{u'}{u} \right]_{t_0}^{t_1}.$$

From this we see that if  $\frac{w'}{w} = \frac{u'}{u}$ , or if  $w = Cu$ , then the second variation is free from the sign of integration; in other words, the second variation is free from the integral sign, if we make any deformation (normal [Art. 113, equation 5]) to the curve) such that the displacement is proportional to the value of any integral of the differential equation  $J = 0$ .

Again, if we deform any one of the family of curves  $G = c$  into a neighboring curve belonging to the family, we have an expression which is also free from the integral sign. For (see Arts. 79 and 81), if we write  $p = \sqrt{x'^2 + y'^2} = \frac{ds}{dt}$ , we have

$$\delta F = G p w_N + \left[ \frac{d}{dt} \left( \xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'} \right) \right]_{t_0}^{t_1},$$

and consequently,

$$\delta^2 F = p w_N \delta G + G \delta(p w_N) + \left[ \frac{d}{dt} \delta \left( \xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'} \right) \right]_{t_0}^{t_1}.$$

Hence, if  $\delta G = 0$ , we have here also

$$\delta^2 I = \left[ \delta \left( \xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'} \right) \right]_{t_0}^{t_1}.$$

It may be shown as follows that the curve  $\delta G = 0$  is one of the family of curves  $G = 0$ . The curves belonging to the family of curves  $G = 0$  are given (Art. 90) by

$$x = \phi(t, \alpha, \beta), \quad y = \psi(t, \alpha, \beta),$$

where  $\alpha$  and  $\beta$  are arbitrary constants. We have a neighboring curve of the family when for  $\alpha, \beta$  we write  $\alpha + \epsilon \alpha', \beta + \epsilon \beta'$ . Then the function  $G$  becomes

$$G + \Delta G = G + \epsilon \delta G + \epsilon^2 ( \quad ) + \dots$$

Hence, when  $\epsilon$  is taken very small, it follows that

$$x = \psi(t, \alpha + \epsilon \alpha', \beta + \epsilon \beta'), \quad y = \psi(t, \alpha + \epsilon \alpha', \beta + \epsilon \beta')$$

is a solution of  $\delta G = 0$ , since it is a solution of  $G + \Delta G = 0$  and of  $G = 0$ .

Now we may always choose normal displacements  $\frac{w}{p}$  which will take us from one of the curves  $G = 0$  to a neighboring curve  $\delta G = 0$ . From this it appears that there is a relation between the differential equations  $\delta G = 0$  and  $J = 0$ .

123. In this connection a discovery made by Jacobi (Crelle's Journal, bd. 17, p. 68) is of great use. He showed that with the integration of the differential equation  $G = 0$ , also that of the differential equation  $J = 0$  is performed. We are then able to derive the general expression for  $u$ , and may determine completely whether and when  $u = 0$ . We shall next derive the general solution of the equation  $J = 0$ , it being presupposed that the differential equation  $G = 0$  admits of a general solution. We derived the first variation in the form

$$\delta I = \int_{t_0}^{t_1} G w dt + \left[ \quad \right]_{t_0}^{t_1}.$$

We may form the second variation by causing in this expression  $G$  alone to vary, and then  $w$  alone, and by adding the results.

It follows that

$$\delta^2 I = \int_{t_0}^{t_1} (\delta G w + G \delta w) dt + \left[ \quad \right]_{t_0}^{t_1}. \quad (i)$$

Since the differential equation  $G = 0$  is supposed satisfied, we have

$$\delta^2 I = \int_{t_0}^{t_1} \delta G w dt + \left[ \quad \right]_{t_0}^{t_1}. \quad (a)$$

We had (Art. 76)

$$\begin{cases} G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right), \\ G_2 = \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right), \end{cases}$$

and also

$$G_1 = y' G, \quad G_2 = -x' G.$$

When in the expression for  $G_1$ , the substitutions

$$\begin{array}{l} x \parallel x + \epsilon \xi, \\ y \parallel y + \epsilon \eta \end{array}$$

are made, we have

$$G_1 + \Delta G_1 = (y' + \epsilon \eta') (G + \Delta G);$$

and since

$$\Delta G_1 = \epsilon \delta G_1 + \epsilon^2 ( \quad ) + \dots,$$

$$\Delta G = \epsilon \delta G + \epsilon^2 ( \quad ) + \dots,$$

it follows that

$$\delta G_1 = y' \delta G + G \eta',$$

and similarly

$$\delta G_2 = -x' \delta G - G \xi'.$$

124. When  $G$  is eliminated from the last two expressions, we have

$$\delta G_1 \xi' + \delta G_2 \eta' = (y' \xi' - x' \eta') \delta G. \quad (ii)$$

On the other hand, it is seen that

$$\begin{aligned} \delta G_1 = & \frac{\partial^2 F}{\partial x^2} \xi + \frac{\partial^2 F}{\partial x \partial y} \eta + \frac{\partial^2 F}{\partial x \partial x'} \xi' + \frac{\partial^2 F}{\partial x \partial y'} \eta' \\ & - \frac{d}{dt} \left\{ \frac{\partial^2 F}{\partial x \partial x'} \xi + \frac{\partial^2 F}{\partial x'^2} \xi' + \frac{\partial^2 F}{\partial x' \partial y} \eta + \frac{\partial^2 F}{\partial x' \partial y'} \eta' \right\}, \end{aligned}$$

an expression which, owing to 2), 3) and 4) of the last Chapter, may be written in the following form :

$$\begin{aligned} \delta G_1 = & \frac{\partial^2 F}{\partial x^2} \xi + \frac{\partial^2 F}{\partial x \partial y} \eta + \frac{\partial^2 F}{\partial x \partial x'} \xi' + \frac{\partial^2 F}{\partial x \partial y'} \eta' \\ & - \frac{dL}{dt} \xi - \frac{dM}{dt} \eta - L \xi' - M \eta' - \frac{d}{dt} \left( F_1 y' \frac{dw}{dt} \right); \end{aligned}$$

and if we take into consideration 3), 4), 6) and 7) of the last Chapter, we may write the above result in the form :

$$\delta G_1 = -y' \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) + y' F_2 w.$$

In an analogous manner, we have

$$\delta G_2 = x' \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) - x' F_2 w.$$

When these values are substituted in (ii), we have

$$\delta G = -\frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) + F_2 w. \quad (b)$$

Hence from (a) we have

$$\delta^2 I = \int_{t_0}^{t_1} \left\{ -\frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) w + F_2 w^2 \right\} dt + \left[ \quad \right]_{t_0}^{t_1}.$$

By the previous method we found the second variation to be [see formula 8) of the last Chapter]

$$\delta^2 I = \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right\} dt + \left[ \quad \right]_{t_0}^{t_1}.$$

These two expressions should agree as to a constant term. The difference of the integrals is

$$D = \int_{t_0}^{t_1} -\frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) w dt - \int_{t_0}^{t_1} F_1 \left( \frac{dw}{dt} \right)^2 dt;$$

but since

$$\int \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) w dt = w F_1 \frac{dw}{dt} - \int F_1 \left( \frac{dw}{dt} \right)^2 dt,$$

it is seen that

$$D = \left[ -w F_1 \frac{dw}{dt} \right]_{t_0}^{t_1}.$$

The formula ( $\delta$ ) is

$$\delta G = F_2 w - \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right).$$

When we compare this with 12<sup>a</sup>) of the preceding Chapter, the differential equation for  $u$ , viz.:

$$0 = F_2 u - \frac{d}{dt} \left( F_1 \frac{du}{dt} \right),$$

it is seen that as soon as we find a quantity  $w$  for which  $\delta G = 0$ , we have a corresponding integral of the differential equation for  $u$ .

125. The total variation of  $G$  is

$$\begin{aligned} \Delta G = G & \left( x + \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots, y + \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots, \right. \\ & x' + \epsilon \xi_1' + \frac{\epsilon^2}{2!} \xi_2' + \dots, y' + \epsilon \eta_1' + \frac{\epsilon^2}{2!} \eta_2' + \dots, \\ & \left. x'' + \epsilon \xi_1'' + \frac{\epsilon^2}{2!} \xi_2'' + \dots, y'' + \epsilon \eta_1'' + \frac{\epsilon^2}{2!} \eta_2'' + \dots \right) \\ & - G(x, y, x', y', x'', y'') = \epsilon \delta G + \frac{\epsilon^2}{2!} \delta^2 G + \dots, \end{aligned}$$

where  $\delta G$ , as found in the preceding article, has the value

$$\delta G = - \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) + F_2 w.$$

Suppose that the equation  $G = 0$  is integrable, and let

$$x = \phi(t, \alpha, \beta), \quad y = \psi(t, \alpha, \beta)$$

be general expressions which satisfy it, where  $\alpha, \beta$  are arbitrary constants of integration. The differential equation  $G = 0$  will be satisfied, if we suppose that  $\alpha$  and  $\beta$ , having arbitrarily fixed values, are increased by two arbitrarily small quantities  $\epsilon \delta \alpha$  and  $\epsilon \delta \beta$ ; that is, the functions

added that  $k$  must be so small that  $F_1 - k$  has the same sign as  $F_1$ , then  $\xi, \eta$  may always be chosen so small that  $|l| < k$ .

The first integral may then be transformed in a manner similar to that in which the integral 8) of Art. 115 was transformed into 14) of Art. 119, and we thus have

$$\begin{aligned} \Delta I = & \int_{t_0}^{t_1} (F_1 - k) \left\{ \frac{dw}{dt} - \frac{d\bar{u}}{dt} \frac{w}{u} \right\}^2 dt \\ & + \int_{t_0}^{t_1} (l + k) \left\{ \left( \frac{dw}{dt} \right)^2 + w^2 \right\} dt, \end{aligned}$$

which shows that  $\Delta I$  for all indefinitely small variations of the curve, which have been brought about under the given assumptions, is positive if  $F_1$  is positive. If  $F_1$  is negative, the same determinations regarding  $k$  remain; only  $k$  must be chosen negative and  $|l| < -k$ . Both integrals on the right of the above equation are then negative, and consequently  $\Delta I$  is itself negative.

We have therefore proved the assertion made above: *If in the interval  $t_0 \dots t_1$  the necessary conditions which were derived from the consideration of the second variation of the integral for the existence of a maximum or a minimum are satisfied, then the sign of the total variation will be the same as the sign of the second variation for all variations of the curve which have been so chosen, that not only the distances between corresponding points on the original curve and the curve subjected to variation are arbitrarily small, but also the directions of both curves at corresponding points deviate from each other by an arbitrarily small quantity.*

It has thus been shown that the three conditions given in Art. 135 are *necessary* for the existence of a maximum or a minimum. A further examination will give a fourth condition (Weierstrass's condition, see Chapter XII) whose fulfillment is also *sufficient*. This condition, if fulfilled, is then decisive, after we have first assured ourselves that the other three conditions are satisfied.

APPLICATION OF THE ESTABLISHED CRITERIA TO THE PROBLEMS I, II, III AND IV, WHICH WERE PROPOSED IN CHAPTER I AND FURTHER DISCUSSED IN CHAPTER VII.

140. PROBLEM I. *The problem of the minimal surface of rotation.*

As the solution of the equation  $G=0$ , we found (Art. 100) the two simultaneous equations of the catenary :

$$1) \quad \begin{cases} x = a + \beta t = \phi(t, a, \beta), \\ y = \beta/2(e^t + e^{-t}) = \psi(t, a, \beta). \end{cases}$$

We have, therefore (Art. 125),

$$2) \quad \begin{cases} \phi'(t) = \beta, & \phi_1(t) = 1, & \phi_2(t) = t, \\ \psi'(t) = \beta/2(e^t - e^{-t}), & \psi_1(t) = 0, & \psi_2(t) = \frac{1}{2}(e^t + e^{-t}); \\ \text{and consequently,} \\ \theta_1(t) = \psi'(t) \phi_1(t) - \phi'(t) \psi_1(t) = \beta/2(e^t - e^{-t}) = y', \\ \theta_2(t) = \psi'(t) \phi_2(t) - \phi'(t) \psi_2(t) = ty' - y. \end{cases}$$

If, now,  $x_0, y_0, x'_0, y'_0$  are the values of  $x, y, x', y'$  which correspond to the value  $t_0$ , then is

$$3) \quad \begin{aligned} \Theta(t, t_0) &= \theta_1(t_0) \theta_2(t) - \theta_2(t_0) \theta_1(t) \\ &= y'_0(ty' - y) - (t_0 y'_0 - y_0) y'; \end{aligned}$$

or, since

$$t = \frac{x-a}{\beta}, \quad t_0 = \frac{x_0-a}{\beta}, \quad \beta = x' = x'_0 \text{ [cf. 2)],}$$

we have

$$4) \quad \Theta(t, t_0) = 1/\beta [y'_0(xy' - yx') - y'(x_0y'_0 - y_0x'_0)].$$

In order to find the point conjugate to  $t_0$  we have to write in this expression for  $x, y, x', y'$  their values in terms of  $t$  and then solve the equation  $\Theta(t, t_0) = 0$ .

To avoid this somewhat complicated calculation, however, we may make use of a geometrical interpretation (Art. 58). The equation of the tangent to the catenary at the point  $x_0, y_0$  is

$$y'_0(X - x_0) - x'_0(Y - y_0) = 0.$$

Therefore, the tangent cuts the  $X$ -axis in the point determined through the equation

$$y'_0 X_0 = x'_0 y_0 - y_0 x'_0.$$

The tangent at any point of the catenary cuts the  $X$ -axis at a point determined by the equation

$$y' X = xy' - yx'.$$

If, now, the point  $x, y$  is to be conjugate to  $x_0, y_0$ , then its coordinates must satisfy 4), which becomes

$$y'_0 y'(X - X_0) = 0.$$

Hence, since  $y'_0$  and  $y'$  do not vanish (Art. 101), we have

$$X = X_0;$$

that is, the conjugate points of the catenary have the property that the tangents drawn through them cut each other on the  $X$ -axis. We thus have an easy geometrical method of determining the point conjugate to any point on the catenary.

Further we have

$$F_1 = \frac{y}{(\sqrt{x'^2 + y'^2})^3},$$

and since  $y$  is always positive, and  $x', y'$  cannot simultaneously vanish, it follows that  $F_1$  is always positive and different from zero and infinity. Hence, the portion of a catenary that is situated between two conjugate points, when rotated about the  $X$ -axis, generates a surface of smallest area (cf. Art. 167).

At the same time in this problem it is seen how small a role the condition regarding  $F_1$  has played in the strenuous proof relative to the existence of a minimum.

141. PROBLEM II. *Problem of the brachistochrone.*

In this problem the expression for  $F_1$  is found to be

$$1) \quad F_1 = \frac{1}{(\sqrt{x'^2 + y'^2})^3} \frac{1}{\sqrt{4gy + a^2}}.$$

We assumed from certain *à priori* reasons that between the points  $A$  and  $B$  of the curve there could be present no cusp (see also Art. 104); that is, no point for which  $x'$  and  $y'$  are both equal to zero simultaneously. For such an arc of the curve  $F_1$  is then always positive and different from zero and infinity, since the quantities under the square root sign are always finite and different from zero (see also Art. 95).

We obtained (Art. 103) the solution of the equation  $G = 0$  in the form

$$2) \quad \begin{cases} x = a + \beta(t - \sin t) = \phi(t, a, \beta), \\ y + a = \beta(1 - \cos t) = \psi(t, a, \beta), \end{cases}$$

where here  $t$  is written in the place of  $\phi$ , and  $a$  in the place of  $-x_0$ , and  $\beta$  instead of  $1/(2c^2)$ ;  $a$  is a given quantity which is determined through the initial velocity.

We consequently have

$$3) \quad \begin{cases} \phi'(t) = \beta(1 - \cos t), \phi_1(t) = 1, \phi_2(t) = t - \sin t; \\ \psi'(t) = \beta \sin t, \psi_1(t) = 0, \psi_2(t) = 1 - \cos t; \\ \theta_1(t) = \beta \sin t, \\ \theta_2(t) = \beta \sin t (t - \sin t) - \beta(1 - \cos t)^2 \\ \quad = 2\beta \sin(t/2) [t \cos(t/2) - 2 \sin(t/2)]; \end{cases}$$

and therefore

$$\Theta(t, t_0) = 4\beta^2 \sin \frac{t_0}{2} \sin \frac{t}{2} \left\{ \cos \frac{t_0}{2} \left( t \cos \frac{t}{2} - 2 \sin \frac{t}{2} \right) - \cos \frac{t}{2} \left( t_0 \cos \frac{t_0}{2} - 2 \sin \frac{t_0}{2} \right) \right\}.$$

With the positions which we have assumed for  $A$  and  $B$  both  $t_0$  and  $t$  are different from  $0$  and  $2\pi$ , and consequently the equation for the determination of the point conjugate to  $t_0$  has the form

$$\cos \frac{t_0}{2} \left( t \cos \frac{t}{2} - 2 \sin \frac{t}{2} \right) - \cos \frac{t}{2} \left( t_0 \cos \frac{t_0}{2} - 2 \sin \frac{t_0}{2} \right) = 0,$$

or

$$4) \quad t - 2 \tan \frac{t}{2} = t_0 - 2 \tan \frac{t_0}{2},$$

which is a transcendental equation for the determination of  $t$ .

We easily see that there is no other real root within the interval  $0 \dots 2\pi$  except  $t=t_0$ , since the derivative of  $t - 2 \tan(t/2)$ , namely,  $1 - \frac{1}{\cos^2(t/2)}$  is negative, so that  $t - 2 \tan(t/2)$  continuously decreases, if  $t$  deviates from  $t_0$ , and can never again take the value  $t_0 - 2 \tan(t_0/2)$ .

Consequently there is no point conjugate to the point  $t_0$  on the arc of the cycloid upon which  $t_0$  lies, and therefore every arc of the cycloid situated between two cusps of this curve has the property that a material point which slides along it from a point  $A$  reaches another point  $B$  of the curve in the shortest time (Art. 168).

In this problem we see that the condition  $F_1 > 0$  was sufficient to establish the existence of a minimum. The case where the initial velocity is zero, and the point  $A$  is situated at one of the cusps will be discussed later (Art. 169).

142. PROBLEM III. *Problem of the shortest line on the surface of a sphere.*

In this problem we find that

$$1) \quad F_1 = \frac{\sin^2 u}{(\sqrt{u'^2 + v'^2 \sin^2 u})^3}.$$

This expression cannot become infinitely large, since  $u'$  and  $v'$  cannot simultaneously vanish.

However, the function  $F_1$  will vanish if  $\sin u = 0$ ; that is,

when  $u=0$  or  $\pi$ . Consequently, in this case, we must so choose the system of coordinates that  $u$  nowhere along the trace of the curve becomes equal to zero or to  $\pi$ . If this has been done, then  $F_1$  for the whole stretch from  $A$  to  $B$  is positive, and does not become zero or infinitely large.

The equation  $G=0$  furnishes the arc of a great circle, whose equations are (see Art. 106):

$$2) \quad \begin{cases} \cos u = \cos c \cos (s-b), \\ \cot (v-\beta) = \sin c \cot (s-b); \\ \text{or,} \\ u = \arccos \{ \cos c \cos (s-b) \} = \phi (s, a, \beta), \\ v = \beta + \arccot \{ \sin c \cot (s-b) \} = \psi (s, a, \beta). \end{cases}$$

Accordingly, we have

$$\phi' (s) = \frac{\cos c \sin (s-b)}{\sqrt{1 - \cos^2 c \cos^2 (s-b)}}, \quad \phi_1 (s) = \frac{\sin s \cos (s-b)}{\sqrt{1 - \cos^2 c \cos^2 (s-b)}},$$

$$\phi_2 (s) = 0, \quad \psi' (s) = \frac{\sin c}{1 - \cos^2 c \cos^2 (s-b)},$$

$$\psi_1 (s) = \frac{-\cos c \sin (s-b) \cos (s-b)}{1 - \cos^2 c \cos^2 (s-b)}, \quad \psi_2 (s) = 1,$$

and consequently

$$\theta_1 (s) = \frac{\cos (s-b)}{\sqrt{1 - \cos^2 c \cos^2 (s-b)}}, \quad \theta_2 (s) = \frac{-\cos c \sin (s-b)}{\sqrt{1 - \cos^2 c \cos^2 (s-b)}}.$$

Hence, since for the point  $A$  we have  $s = s_0 = 0$ , it follows that

$$3) \quad \Theta (s, s_0) = - \frac{\cos c \sin s}{\sqrt{1 - \cos^2 c \cos^2 b} \sqrt{1 - \cos^2 c \cos^2 (s-b)}}.$$

Therefore, in order to find the point conjugate to the point  $s_0 = 0$ , we have to solve the equation  $\Theta (s, s_0) = 0$  with respect to  $s$ .

Since the denominator of 3) cannot become infinite, the conjugate point is to be determined from the equation  $\sin s = 0$ . We consequently have  $s = \pi$  as the point conjugate to  $s = 0$ ; that is, the point conjugate to  $A$  is the other end of the diameter of the circle drawn through  $A$ .

Hence the arc of a great circle through the points  $A$  and  $B$ , measured in a direction fixed as positive, is the shortest distance upon the surface of the sphere *only* when these points are not at a distance of  $180^\circ$  or more from each other, a result which is of itself geometrically clear.

We may remark that the condition that  $F_1$  cannot vanish is clearly in this case unnecessary; since the arc of a great circle possesses the property of a minimum independently of the choice of the system of coordinates with respect to which  $F_1$ , say, at some point of the curve vanishes.

143. From the figure in Art. 107 it is clear that when  $A$  is the pole of the sphere, the family of curves passing through  $A$  and satisfying the differential equation  $G = 0$  (*i. e.*, arcs of great circles) intersect again only at the other pole. In the next Chapter it will appear that the two poles are conjugate points. This, together with what was given in the preceding article, may be taken as a proof that the arcs of great circles can meet only at opposite poles.

144. PROBLEM IV. *Problem of the surface offering the least resistance.*

In this problem let us write (Art. 110)

$$a = -C/2, \quad \beta = -C_1,$$

so that

$$x = a[t + 2t^{-1} + t^{-3}] = \phi(t, a, \beta),$$

$$y = a[\log t + t^{-2} + \frac{3}{4}t^{-4}] - \beta = \psi(t, a, \beta).$$

Hence,

$$\phi'(t) = x', \quad \phi_1(t) = x/a, \quad \phi_2(t) = 0,$$

$$\psi'(t) = y', \quad \psi_1(t) = (y + \beta)/a, \quad \psi_2(t) = -1,$$

$$\left. \begin{aligned} \theta_1(t) &= y' x/a - x'(y + \beta)/a, \\ \theta_2(t) &= x', \end{aligned} \right\}$$

and

$$\Theta(t, t_0) = \frac{x'(y'_0 x_0 - x'_0 y_0) - x'_0(y'x - x'y)}{a}.$$

Now the tangent to the curve at any point  $x_0, y_0$  is

$$y'_0(X - x_0) - x'_0(Y - y_0) = 0,$$

and the intercept on the  $Y$ -axis is

$$x'_0 Y_0 = x'_0 y_0 - y'_0 x_0.$$

The tangent to the curve at any point  $x, y$  cuts the  $Y$ -axis where

$$x'Y = x'y - y'x.$$

We therefore have for the determination of the point conjugate to  $x_0, y_0$  the equation

$$x'x'_0 Y_0 = x'_0 x'Y, \quad \text{or} \quad Y_0 = Y.$$

As in Art. 140, this gives an easy geometrical construction for conjugate points.

$$\bar{x} = \varphi(t, a + \epsilon \delta a, \beta + \epsilon \delta \beta) = \phi(t, a, \beta) + \epsilon \left( \frac{\partial \phi}{\partial a} \delta a + \frac{\partial \phi}{\partial \beta} \delta \beta \right) + \epsilon^2 ( \quad ),$$

$$\bar{y} = \psi(t, a + \epsilon \delta a, \beta + \epsilon \delta \beta) = \psi(t, a, \beta) + \epsilon \left( \frac{\partial \psi}{\partial a} \delta a + \frac{\partial \psi}{\partial \beta} \delta \beta \right) + \epsilon^2 ( \quad )$$

are also solutions of  $G = 0$ .

126. Now choose the variation of the curve (Art. 111) in such a way that

$$\bar{x} = x + \epsilon \xi_1 + \frac{\epsilon^2}{2!} \xi_2 + \dots,$$

$$\bar{y} = y + \epsilon \eta_1 + \frac{\epsilon^2}{2!} \eta_2 + \dots;$$

and, whatever be the values of  $\delta a$  and  $\delta \beta$ , we determine  $\xi_1, \xi_2, \eta_1, \eta_2$ , etc., by the relations:

$$\left. \begin{aligned} \xi_1 &= \frac{\partial \phi}{\partial a} \delta a + \frac{\partial \phi}{\partial \beta} \delta \beta, \\ \eta_1 &= \frac{\partial \psi}{\partial a} \delta a + \frac{\partial \psi}{\partial \beta} \delta \beta. \end{aligned} \right\} \quad (iii)$$

For all values of  $a$  and  $\beta$  the differential equation  $G = 0$  is satisfied; hence, the values of  $\xi_1, \eta_1$ , etc., just written, when substituted in  $\Delta G$  above must make the right-hand side of that equation vanish identically, and consequently also  $\delta G$ . Hence, the corresponding normal displacement  $w = y' \xi_1 - x' \eta_1$  transforms one of the system of curves  $G = 0$  to another one of the same system.

Since  $\delta a$  and  $\delta \beta$  are entirely arbitrary, the coefficients of  $\delta a$  and  $\delta \beta$  must each vanish in the expansion of  $\Delta G$  above. Owing to (iii)  $w = y' \xi_1 - x' \eta_1$  becomes

$$w = \left( y' \frac{\partial \phi}{\partial a} - x' \frac{\partial \psi}{\partial a} \right) \delta a + \left( y' \frac{\partial \phi}{\partial \beta} - x' \frac{\partial \psi}{\partial \beta} \right) \delta \beta.$$

Writing this value of  $w$  in the equation  $\delta G = 0$ , we have

$$\begin{aligned} & - \frac{d}{dt} \left\{ F_1 \frac{d}{dt} \left[ \left( y' \frac{\partial \phi}{\partial a} - x' \frac{\partial \psi}{\partial a} \right) \delta a + \left( y' \frac{\partial \phi}{\partial \beta} - x' \frac{\partial \psi}{\partial \beta} \right) \delta \beta \right] \right\} \\ & + F_2 \left[ \left( y' \frac{\partial \phi}{\partial a} - x' \frac{\partial \psi}{\partial a} \right) \delta a + \left( y' \frac{\partial \phi}{\partial \beta} - x' \frac{\partial \psi}{\partial \beta} \right) \delta \beta \right] = 0. \end{aligned}$$

By equating the coefficients of  $\delta a$  and  $\delta \beta$  respectively to zero, we have the two equations:

$$1) \quad -\frac{d}{dt} \left\{ F_1 \frac{d}{dt} \theta_\nu(t) \right\} + F_2 \theta_\nu(t) = 0, \\ (\nu = 1, 2),$$

where, for brevity, we have written

$$2) \left\{ \begin{array}{l} \frac{\partial \phi(t)}{\partial t} = \phi'(t), \quad \frac{\partial \phi}{\partial a} = \phi_1(t), \quad \frac{\partial \phi}{\partial \beta} = \phi_2(t), \\ \frac{\partial \psi(t)}{\partial t} = \psi'(t), \quad \frac{\partial \psi}{\partial a} = \psi_1(t), \quad \frac{\partial \psi}{\partial \beta} = \psi_2(t), \\ \theta_1(t) = \psi'(t) \phi_1(t) - \phi'(t) \psi_1(t), \\ \theta_2(t) = \psi'(t) \phi_2(t) - \phi'(t) \psi_2(t). \end{array} \right.$$

It is seen at once that  $\theta_1(t)$  and  $\theta_2(t)$  are the solutions of the differential equation

$$\frac{d}{dt} \left( F_1 \frac{du}{dt} \right) - F_2 u = 0.$$

Hence it is seen that the general solution of the differential equation for  $u$  is had from the integrals of the differential equation  $G=0$ , through simple differentiation.

127. We have next to prove that the two solutions  $\theta_1(t)$  and  $\theta_2(t)$  are independent of each other. In order to make this proof as simple as possible, let  $x$  be written for the arbitrary quantity  $t$ .

Then the expressions  $x = \phi(t, a, \beta)$ ,  $y = \psi(t, a, \beta)$ , etc., become

$$x = x, \quad y = \psi(x, a, \beta), \\ \phi' = 1, \quad \phi_1 = 0, \quad \phi_2 = 0, \quad \psi' = \frac{dy}{dx}, \\ \theta_1 = -\psi_1, \quad \theta_2 = -\psi_2.$$

If  $\theta_1$  and  $\theta_2$  are linearly dependent upon each other, we must have

$$\theta_2 = \text{constant } \theta_1,$$

from which it follows, at once, that

$$\theta_1 \theta_2' - \theta_2 \theta_1' = 0,$$

where the accents denote differentiation with respect to  $x$ ; or,

$$\psi_1 \psi_2' - \psi_2 \psi_1' = 0.$$

(On the other hand,  $y = \psi(x, \alpha, \beta)$  is the complete solution of the differential equation, which arises out of  $G_2 = -x' G = 0$ , when  $x$  is written for  $t$ ; that is, of

$$\frac{d}{dx} \left( \frac{\partial F}{\partial \frac{dy}{dx}} \right) - \frac{\partial F}{\partial y} = 0;$$

but here  $\alpha$  and  $\beta$  are two arbitrary independent constants, and consequently  $\psi$  and  $\psi' = \frac{d\psi}{dx}$  are independent of each other with respect to  $\alpha$  and  $\beta$ , so that the determinant

$$\psi_1 \psi_2' - \psi_2 \psi_1'$$

is different from zero. Consequently  $\theta_1$  and  $\theta_2$  are independent of each other, since the contrary assumption stands in contradiction to the result just established. Hence, the general solution of the differential equation  $J=0$ , is of the form

$$u = c_1 \theta_1(t) + c_2 \theta_2(t),$$

where  $c_1$  and  $c_2$  are arbitrary constants.

128. Following the methods of Weierstrass we have just proved the assertion of Jacobi; since, as soon as we have the complete integral of  $G=0$ , it is easy to express the complete solution of the differential equation  $J=0$ .

The constants  $c_1$  and  $c_2$  may be so determined that  $u$  vanishes on a definite position  $t'$ , which may lie somewhere on the curve before we get to  $t_1$ . This may be effected by writing

$$c_1 = -\theta_2(t'), \quad c_2 = \theta_1(t').$$

The solution of the equation  $J=0$  becomes

$$3) \quad u = \theta_1(t') \theta_2(t) - \theta_2(t') \theta_1(t) = \Theta(t, t').$$

It may turn out that  $\Theta(t, t')$  vanishes for no other value of  $t$ ; but it may also happen that there are other positions than  $t'$  at which  $\Theta(t, t')$  becomes zero. If  $t''$  is the first zero position of  $\Theta(t, t')$  which follows  $t'$ , then  $t''$  is called *the conjugate point to  $t'$* .

Since  $t'$  has been arbitrarily chosen, we may associate with every point of the curve a second point, its conjugate. This being premised, we come to the following theorem, also due to Jacobi:

*If within the interval  $t_0 \dots t_1$  there are no two points which are conjugate to each other in the above sense, then it is possible so to determine  $u$  that it satisfies the differential equation  $J=0$ , and nowhere vanishes within the interval  $t_0 \dots t_1$ .*

129. Let the point  $t = t'$  be a zero position of the function

$$u = \Theta(t, t'),$$

and let  $t''$  be a conjugate point to  $t'$ , then  $\Theta(t, t')$  will not again vanish within the interval  $t' \dots t''$ . Take in the neighborhood of the point  $t'$  a point  $t' + \tau$ , where  $\tau > 0$ , then the point which is conjugate to  $t' + \tau$  can lie only on the other side of  $t''$ . This may be shown as follows:

If  $u = \Theta(t, t')$  is a solution of the equation

$$F_1 \frac{d^2 u}{dt^2} + \frac{dF_1}{dt} \frac{du}{dt} - F_2 u = 0,$$

then is

$$\bar{u} = \Theta(t, t' + \tau)$$

a solution of the same equation; that is, of

$$F_1 \frac{d^2 \bar{u}}{dt^2} + \frac{dF_1}{dt} \frac{d\bar{u}}{dt} - F_2 \bar{u} = 0,$$

since  $\bar{u}$  differs from  $u$  only through another choice of the arbitrary constants  $c_1$  and  $c_2$ .

If  $\tau$  is chosen sufficiently small, then  $\Theta(t' + \tau, t')$  is different from zero, and consequently also  $\Theta(t', t' + \tau) \neq 0$ .

Eliminate  $F_2$  from the two equations above, and we have

$$4) \quad F_1 \left( u \frac{d^2 \bar{u}}{dt^2} - \bar{u} \frac{d^2 u}{dt^2} \right) + \frac{dF_1}{dt} \left( u \frac{d\bar{u}}{dt} - \bar{u} \frac{du}{dt} \right) = 0.$$

Now write

$$5) \quad u \frac{d\bar{u}}{dt} - \bar{u} \frac{du}{dt} = v,$$

and the above equation becomes

$$6) \quad \frac{dv}{v} = - \frac{dF_1}{F_1},$$

which, when integrated, is

$$7) \quad v = u \frac{d\bar{u}}{dt} - \bar{u} \frac{du}{dt} = + \frac{C}{F_1}.$$

The constant  $C$  in this expression cannot vanish, for, in that case,

$$u = \text{const. } \bar{u},$$

or

$$\Theta(t, t') = \text{const. } \Theta(t, t' + \tau).$$

Since, however,  $\Theta(t, t')$  vanishes for  $t = t'$ , it results from the above that  $\Theta(t', t' + \tau) = 0$ , which is contrary to the hypothesis, and consequently  $C$  cannot vanish.

It is further assumed that  $F_1$  does not change its sign or become zero within the interval  $t_0 \dots t_1$ . If  $F_1$  vanishes without a transition from the positive to the negative or *vice versa* within the stretch  $t_0 \dots t_1$ , then in general no further deductions can be drawn, and a special investigation has to be made for each particular case.

In the first case, however,  $v$  has a finite value, and the equation 7), when divided through by  $u^2$ , becomes

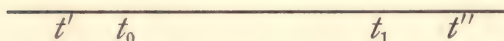
$$\frac{u \frac{d\bar{u}}{dt} - \bar{u} \frac{du}{dt}}{u^2} = \frac{d \frac{\bar{u}}{u}}{dt} = \frac{C}{F_1 u^2},$$

an expression, which, when integrated, is

$$\bar{u} = Cu \int_{t'+\tau}^{t''} \frac{dt}{F_1 u^2}.$$

Since the function  $u$  does not vanish between  $t'$  and  $t''$ , it follows from the last expression that  $\bar{u}$  cannot vanish between the limits  $t'+\tau$  and  $t''$ . Accordingly, if there is a point conjugate to  $t'+\tau$ , it cannot lie before  $t''$ . If, therefore, we choose a point  $t'''$  before  $t''$  and as close to it as we wish, then  $u^*$  will certainly not vanish within the interval  $t'+\tau \dots t'''$ .

If  $t'$  is a point situated immediately before  $t_0$ , and if we determine the point  $t''$  conjugate to  $t'$ , and choose a point  $t_1$  before  $t''$  and as near to it as we wish, then from the preceding it is clear that no points conjugate to each other lie within



the interval  $t_0 \dots t_1$ , the boundaries excluded. We may then, as shown above, find a function  $u$ , which satisfies the differential equation  $J=0$  and which vanishes neither on the limits nor within the interval  $t_0 \dots t_1$ . The transformation of Art. 117 is therefore admissible, and the sign of  $\delta^2 I$  depends only upon the sign of  $F_1$ .

130. We may investigate a little more closely the relation of Art. 120, where

$$u_2 \frac{du_1}{dt} - u_1 \frac{du_2}{dt} = \frac{C}{F_1}.$$

In the interval under consideration, boundaries included, we assume that  $F_1$  does not become zero or infinite, and consequently retains the same sign. Further, the constant  $C$  has always the same value and is different from zero, since  $u_1$  and  $u_2$  are linearly independent.

It follows at once that  $\frac{du_1}{dt}$  cannot be zero at the same time that  $u_1$  is zero; for then  $C$  would be zero contrary to our hypothesis.

Owing to the form

$$\frac{d}{dt} \left( \frac{u_1}{u_2} \right) = \frac{1}{u_2^2} \frac{C}{F_1},$$

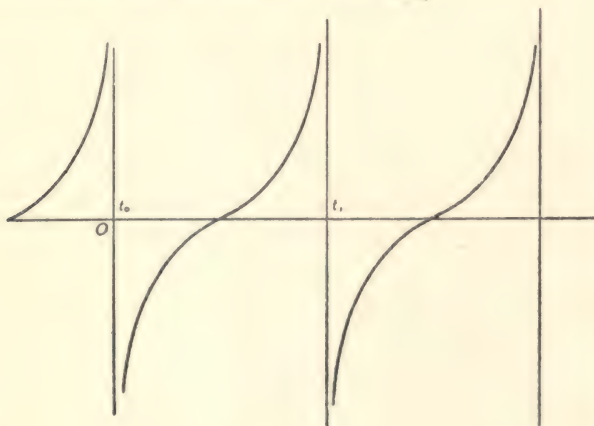
it is clear that  $\frac{d}{dt} \left( \frac{u_1}{u_2} \right)$  has the same sign as  $\frac{C}{F_1}$ . We may take this sign positive, since otherwise owing to the expression

$$u_1 \frac{du_2}{dt} - u_2 \frac{du_1}{dt} = \frac{C}{F_1}$$

we would have  $\frac{d}{dt} \left( \frac{u_2}{u_1} \right)$  positive. We may assume then that the indices have been placed upon the  $u$ 's, so that  $\frac{u_1}{u_2}$  is always on the increase with increasing  $t$ .

The ratio  $\frac{u_1}{u_2}$  will become infinite for the zero values of  $u_2$  (see Art. 120). Since this quotient is always increasing with increasing values of  $t$ , the trace of the corresponding curve must pass through  $+\infty$ , and return again (if it does return) from  $-\infty$ . Values of  $t$ , for which this quotient has the same value, may be called *congruent*.

It is evident, as shown in the accompanying figure, that such values are equi-distant from two values of  $t$ , say  $t_0$  and  $t_1$ , which make  $u_2 = 0$ . The abscissæ are values of  $t$ , and the ordinates are the corresponding values of the ratio  $\frac{u_1}{u_2}$ .



131. To summarize: We have supposed the cases excluded in which  $F_1$  is zero along the curve under consideration. If this function were zero at an isolated point of the curve, it would be a limiting case of what we have considered. If it were zero along a

stretch of this curve, we should have to consider variations of the third order, and would have, in general, neither a maximum nor a minimum value unless this variation also vanished, leaving us to investigate variations of the fourth order. We exclude these cases from the present treatment, and suppose also that  $F_1$  and  $F_2$  are everywhere finite along our curve (otherwise the expression for the second variation, viz.—

$$\int (F_1 w'^2 + F_2 w^2) dt,$$

would have no meaning).

We also derived in Art. 124 the variation of  $G$  in the form

$$\delta G = F_2 w - \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right),$$

and when this is compared with the differential equation

$$12^a) \quad 0 = F_2 u - \frac{d}{dt} \left( F_1 \frac{du}{dt} \right) \quad (\text{see Art. 118}),$$

it is seen that if an integral  $u$  of the differential equation  $12^a$ ) vanishes for any value of  $t$ , the corresponding integral  $w$  of the equation  $\delta G = 0$  vanishes for the same value of  $t$ .

In Art. 126 we had

$$w = y'\xi_1 - x'\eta_1 = \delta\alpha\theta_1(t) + \delta\beta\theta_2(t),$$

where the displacement  $\xi_1, \eta_1$  takes us from a point of the curve  $G = 0$  to a point of the curve  $\delta G = 0$ . Consequently the normal displacement  $w_N$  can be zero only at a point where the curves  $G = 0$  and  $\delta G = 0$  intersect.

At such a point we must have

$$\delta\alpha\theta_1(t) + \delta\beta\theta_2(t) = 0.$$

When one of the family of curves  $G = 0$  has been selected, the two associated constants  $\alpha$  and  $\beta$  are fixed. These are the constants that occur in  $\theta_1(t)$  and  $\theta_2(t)$ . If, further, the curve passes through a fixed point  $P_0$ , the variable  $t$  is determined, and consequently the functions  $\theta_1(t)$  and  $\theta_2(t)$  are definitely determined, so that the

ratio  $\delta\alpha:\delta\beta$  is definitely known from the above relation. There may be a second point at which the curves  $G=0$  and  $\delta G=0$  intersect. This point is the point conjugate to  $P_0$  (see Art. 128).

132. The geometrical significance of these conjugate points is more fully considered in Chapter XI. Writing the second variation in the form

$$\delta^2 I = \int_{t_0}^{t_1} F_1 w^2 \left( \frac{w'}{w} - \frac{u'}{u} \right)^2 dt,$$

we see that the possibility of  $\frac{w'}{w} - \frac{u'}{u} = 0$  is when  $u=Cw$ . Now  $w$  is zero at both of the end-points of the curve, since at these points there is no variation, but  $u$  is equal to zero at  $P_1$  only when  $P_1$  is conjugate to  $P_0$ . Hence, unless the two curves  $G=0$  and  $\delta G=0$  intersect again at  $P_1$ ,  $u$  is not equal to zero at  $P_1$ , and consequently

$$\left( \frac{w'}{w} - \frac{u'}{u} \right)^2 \neq 0.$$

*In this case, if  $F_1$  has a positive sign throughout the interval  $t_0 \dots t_1$ , there is a possibility of a minimum value of the integral  $I$ , and there is a possibility of a maximum value when  $F_1$  has a negative sign throughout this interval.*

133. Next, let  $P_1$  be conjugate to  $P_0$ , so that at both of the limits of integration we have  $u=0=w$ . We may then take  $u=w$  at all other points of the curve, so that consequently

$$\delta^2 I = \int_{t_0}^{t_1} F_1 w^2 \left( \frac{w'}{w} - \frac{u'}{u} \right)^2 dt = 0.$$

We cannot then say anything regarding a maximum or a minimum until we have investigated the variations of a higher order.\*

\* It is sometimes possible to establish the *existence* or the *non-existence* of a maximum or a minimum by other methods; for example, the *non-existence* of a minimum is seen in Case II of Art. 58. In a very *instructive* paper (Trans. of the Am. Math. Soc., Vol. II, p. 166) Prof. Osgood has shown that there *is* a minimum in the case of the geodesics on an ellipsoid of revolution (due to the fact that the curve must lie on the ellipsoid). Prof. Osgood says (p. 166) that Kneser's Theorem "to the effect that there is *not* a minimum" is in general true. It seems that each separate case must be examined for itself, and in general nothing can be said regarding a maximum or a minimum.

Next, suppose that a pair of conjugate points are situated between  $P_0$  and  $P_1$ , and let these points be  $P'$  and  $P''$ . We may then make a displacement of the curve so that

$$w = kw \text{ from } P_0 \text{ to } P',$$

$$w = u + kw \text{ from } P' \text{ to } P'' \text{ and}$$

$$w = kw \text{ from } P'' \text{ to } P_1,$$

where  $k$  is an indeterminate constant. The quantity  $w$  is subjected only to the condition that it must be zero at  $P_0$  and  $P_1$ , and  $u$  must be a solution of the differential equation  $J = 0$ , and is zero at the conjugate points  $P'$  and  $P''$ .

The second variation takes the form

$$\begin{aligned} \delta^2 I &= k^2 \int_{t_0}^{t'} (F_1 w'^2 + F_2 w^2) dt \\ &+ \int_{t'}^{t''} \{ (F_1 u'^2 + F_2 u^2) + 2k (F_1 u' w' + F_2 u w) \\ &\quad + k^2 (F_1 w'^2 + F_2 w^2) \} dt \\ &+ k \int_{t''}^{t_1} (F_1 w'^2 + F_2 w^2) dt. \end{aligned}$$

In the preceding article we saw (cf. also Art. 117) that

$$\int_{t'}^{t''} (F_1 u'^2 + F_2 u^2) dt = 0,$$

and we may therefore write  $\delta^2 I$  in the form

$$\delta^2 I = 2k \int (F_1 u' w' + F_2 u w) dt + k^2 M,$$

where  $M$  is a finite quantity.

The integral

$$\int_{t'}^{t''} (F_1 u' w' + F_2 u w) dt$$

may be written

$$\int_{t'}^{t''} \left\{ -\frac{d}{dt} (F_1 u') + F_2 u \right\} w dt + [F_1 u' w]_{t'}^{t''},$$

and since, in virtue of the formula 12<sup>a</sup>) of Art. 118, the expression under this latter integral sign is zero, it follows that

$$\delta^2 I = 2k [F_1 u' w]_{t'}^{t''} + k^2 M.$$

Further, by hypothesis,  $F_1$  retains the same sign within the interval  $t' \dots t''$ , and does not become zero within or at these limits, the function  $u'$  is different from zero at the limits (Arts. 130 and 152), and of opposite sign at these limits, since  $u$ , always retaining the same sign, leaves the value zero at one limit and approaches it at the other limit. Consequently  $[F_1 u']$  is finite and of opposite signs at the two points  $P'$  and  $P''$ , and it remains only that  $w$  be chosen finite and with the same sign, so that  $[F_1 u' w]_{t'}^{t''}$  be different from zero. Hence by the proper choice of  $k$  we may effect displacements for which  $\delta^2 I$  is positive, and also those for which it is negative.

*Hence when our interval includes (not, however, both as extremities) a pair of conjugate points, we have definitely established that the curve in question can give rise to neither a maximum nor a minimum.*

The above semi-geometrical proof is due to a note given by Prof. Schwarz at Berlin (1898-99); see also Leçon 1<sup>re</sup> of a course of Lectures given by Prof. Picard at Paris (1899-1900) on "*Equations aux dérivées partielles.*"

## CHAPTER X.

THE CRITERIA THAT HAVE BEEN DERIVED UNDER THE ASSUMPTION OF CERTAIN SPECIAL VARIATIONS ARE ALSO SUFFICIENT FOR THE ESTABLISHMENT OF THE FORMULÆ HITHERTO EMPLOYED.

134. The methods which we have followed would indicate that the whole process of the Calculus of Variations is a process of progressive exclusion. We first exclude curves for which  $G$  is different from zero and limit ourselves to curves which satisfy the differential equation  $G = 0$ . From these latter curves we exclude all those along which  $F_1$  does not retain the same sign. If, for any curves not yet excluded,  $F_1 = 0$  at isolated points, we have simply a limiting case among those to which our conclusions apply. If  $F_1 = 0$  for a stretch of curve not excluded by the above condition, we have to subject the curve to additional consideration in which the third and higher variations must be investigated. We further exclude all curves, in which conjugate points are found situated between the limits of integration, as being impossible generators of a maximum or a minimum. The cases in which no such pairs of points are to be found, or where such points are the limits of integration, require further investigation. This leads us to a fourth condition, a condition due to Weierstrass, which is discussed in Chapter XII. In this process of exclusion let us next see whether the variations admitted are sufficient for the general treatment under consideration.

135. As necessary conditions for the appearance of a maximum or a minimum, the following theorems have been established:

1)  $x, y$  as functions of  $t$  must be determined in such a manner that they satisfy the differential equation  $G=0$ .

2) Along the curve that has been so determined the function  $F_1$  cannot be positive for a maximum nor can it be negative for a minimum; moreover, the case that  $F_1=0$  at isolated points or along a certain stretch, cannot in its generality be treated, but the problems that thus arise must be subjected to special investigation.

3) The integration may extend at most from a point to its conjugate point, but not beyond this point.

The last two conditions, which were derived from the consideration of the second variation, require certain limitations. On the one hand, a proof has to be established that the sign of  $\Delta I$  is in reality the same as the sign of  $\delta^2 I$ , if we choose for  $\xi, \eta$ , etc., the most general variations of all those special variations, for which the developments hitherto made were true; it then remains to investigate whether and how far the criteria which have been established remain true for the case where the curve varies quite arbitrarily.

136. We return to the proof of the theorem proposed in the preceding article. We have, in the case of the investigations hitherto made, always assumed that  $\xi, \eta, \xi', \eta'$  were sufficiently small quantities, since only under this assumption can we develop the right-hand side of

$$\Delta I = \int_{t_0}^{t_1} \{F(x+\xi, y+\eta, x'+\xi', y'+\eta') - F(x, y, x', y')\} dt$$

in powers of these quantities. This means not only that the curve which has been subjected to variation must lie indefinitely near the original curve, but also that the direction of the two curves can differ only a little from each other at corresponding points. We retain the same assumptions, and limit ourselves always to special variations.

We shall first prove that for all these variations the sign of  $\Delta I$  and that of  $\delta^2 I$  agree, so that for these variations the criteria already found are also sufficient. However, we no longer assume that the variations are expressible in the form  $\epsilon \xi, \epsilon \eta$ , where  $\epsilon$  denotes a sufficiently small quantity.

Since the curvature of the original curve does not become infinitely large at any point (see Art. 95), and since further the original curve and the curve which has been subjected to variation deviate only a little from each other at corresponding points both in their position and the direction of their tangents, it follows that with each point of the original curve is associated the point of the curve that has been varied, in which the latter curve is cut by the normal drawn through the point on the first curve.

The equation of the normal at the point  $x, y$  is

$$(X - x) x' + (Y - y) y' = 0;$$

and from the remarks just made, the point  $x + \xi, y + \eta$  is to lie on this normal, so that

$$\xi x' + \eta y' = 0.$$

If we consider this equation in connection with the definition of  $w$ :

$$w = \xi y' - \eta x',$$

it follows that the variations may be represented in the form

$$1) \quad \xi = \frac{w}{x'^2 + y'^2} y', \quad \eta = - \frac{w}{x'^2 + y'^2} x'.$$

In these expressions  $\frac{w}{x'^2 + y'^2}$  is an indefinitely small quantity, since  $x'$  and  $y'$  cannot both vanish at the same time (Art. 95), and it varies in a continuous manner with  $t$ . Likewise the derivative of this quantity with respect to  $t$  is an indefinitely small quantity which, however, may not be everywhere continuous.

Under the assumption that  $\xi, \eta, \xi' = \frac{d\xi}{dt}, \eta' = \frac{d\eta}{dt}$  are sufficiently small quantities, we may develop the total variation of the integral

$$\Delta I = \int_{t_0}^{t_1} [F(x + \xi, y + \eta, x' + \xi', y' + \eta') - F(x, y, x', y')] dt$$

with respect to the powers of  $\xi, \eta, \xi', \eta'$ ; and, if we make use of Taylor's Theorem in the form

$$F(x + \xi_1, \dots, x_n + \xi_n) = F(x_1, \dots, x_n) + \sum_i \frac{\partial F}{\partial x_i} \xi_i + \int_0^1 (1 - \epsilon) d\epsilon \sum_{i,j} [F_{i,j}(x_1 + \epsilon \xi_1, \dots, x_n + \epsilon \xi_n) \xi_i \xi_j],$$

where  $F_{i,j} = \frac{\partial^2 F}{\partial x_i \partial x_j}$ , we have, since the terms of the first dimension vanish, a development of the form

$$2) \quad \Delta I = \int_{t_0}^{t_1} \int_0^1 (1 - \epsilon) [F_{1,1}(x + \epsilon \xi, y + \epsilon \eta, x' + \epsilon \xi', y' + \epsilon \eta') \xi^2 + \dots] d\epsilon dt.$$

137. If we further develop  $F_{1,1}$ , etc., with respect to powers of  $\epsilon$ , it is found that the aggregate of terms that do not contain  $\epsilon$  is identical with  $\delta^2 F$  which was obtained in Chapter VIII.

Integrating with respect to  $\epsilon$ , we may represent the other terms as a quadratic form in  $\xi, \eta, \xi', \eta'$ , whose coefficients also contain these quantities and in such a way that they become indefinitely small with these quantities.

Next, writing in  $\Delta I$  the values of  $\xi, \eta$  given in 1) and the following values of  $\xi', \eta'$  also derived from 1):

$$3) \quad \begin{cases} \xi' = \frac{y'}{x'^2 + y'^2} \frac{dw}{dt} + w \frac{d}{dt} \left( \frac{y'}{x'^2 + y'^2} \right), \\ \eta' = -\frac{x'}{x'^2 + y'^2} \frac{dw}{dt} - w \frac{d}{dt} \left( \frac{x'}{x'^2 + y'^2} \right), \end{cases}$$

we have

$$4) \quad \Delta I = \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right\} dt \\ + \int_{t_0}^{t_1} \left\{ f w^2 + 2 g w \frac{dw}{dt} + h \left( \frac{dw}{dt} \right)^2 \right\} dt,$$

where  $f, g, h$  denote functions which still contain  $w$  and  $\frac{dw}{dt}$ , and in such a way that they become indefinitely small at the same time as these quantities.

138. After a known theorem\* in quadratic forms,

$$f w^2 + 2 g w \frac{dw}{dt} + h \left( \frac{dw}{dt} \right)^2$$

may always, through linear substitutions not involving imaginaries, be brought to the form

$$f_1 u_1^2 + f_2 u_2^2,$$

in such a way that at the same time the relation

$$u_1^2 + u_2^2 = w^2 + \left( \frac{dw}{dt} \right)^2$$

is true, and where  $f_1$  and  $f_2$  are roots of the quadratic equation in  $x$ :

$$(f - x)(h - x) - g^2 \equiv x^2 - x(f + h) + fh - g^2 = 0.$$

Since the coefficients in this equation become simultaneously small with  $w$  and  $\frac{dw}{dt}$ , the same must also be true of  $f_1$  and  $f_2$ , the roots of this equation.

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\* Such substitutions are called by Cayley *orthogonal* (Crelle, bd. 32, p. 119); see also Euler, Nov. Comm. Petrop., 15, p. 275; 20, p. 217; Cauchy, Exerc. de Math., 4, p. 140; Jacobi, Crelle, bd. 12, p. 7; bd. 30, p. 46; Baltzer, Theorie und Anwendungen der Determinanten, 1881, p. 187; Rodrigues, Liouv. Journ., t. 5, p. 405; Hesse, Crelle, bd. 57, p. 175.

If  $l$  is the mean value between  $f_1$  and  $f_2$ , which also becomes indefinitely small with  $w$  and  $\frac{dw}{dt}$ , we may bring the expression

$$f_1 u_1^2 + f_2 u_2^2$$

to the form

$$f_1 u_1^2 + f_2 u_2^2 = l (u_1^2 + u_2^2) = l \left\{ w^2 + \left( \frac{dw}{dt} \right)^2 \right\},$$

and consequently we have for  $\Delta I$  the expression

$$\Delta I = \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right\} dt + \int_{t_0}^{t_1} l \left\{ w^2 + \left( \frac{dw}{dt} \right)^2 \right\} dt,$$

or finally

$$\Delta I = \int_{t_0}^{t_1} \left\{ (F_1 + l) \left( \frac{dw}{dt} \right)^2 + (F_2 + l) w^2 \right\} dt;$$

and thus we have for  $\Delta I$  the same form as we had before for  $\delta^2 I$  (Art. 115).

139. We assume now that the necessary conditions for the existence of a maximum or a minimum are satisfied; that therefore along the whole curve  $G=0$ , the function  $F_1$  is different from zero or infinity, and always retains the same sign; that a function  $u$  may be determined which satisfies the equation

$$\frac{d}{dt} \left( F_1 \frac{du}{dt} \right) - F_2 u = 0,$$

and nowhere vanishes within the interval  $t_0 \dots t_1$  or upon the boundaries of this interval.

If we therefore understand by  $k$  a positive quantity, and write

$$l = -k + l + k,$$

then the expression for  $\Delta I$  above becomes

$$\Delta I = \int_{t_0}^{t_1} \left\{ (F_1 - k) \left( \frac{dw}{dt} \right)^2 + (F_2 - k) w^2 \right\} dt$$

$$+ \int_{t_0}^{t_1} (l + k) \left\{ \left( \frac{dw}{dt} \right)^2 + w^2 \right\} dt.$$

If  $k$  is given a fixed value, then we may choose  $\xi, \eta$  so small that the absolute value of the quantity  $l$  that depends upon them is less than  $k$ . The quantity  $k + l$  is therefore positive, and consequently also the second integral of the above expression. We have yet to show that the first integral is also positive, if  $F_1 > 0$ .

After a known theorem in differential equations it is always possible, as soon as the equation

$$\frac{d}{dt} \left( F_1 \frac{du}{dt} \right) - F_2 u = 0$$

is integrated through a continuous function  $u$  of  $t$ , which within and on the boundaries of the interval  $t_0 \dots t_1$  nowhere vanishes, also to integrate the differential equation

$$\frac{d}{dt} \left[ (F_1 - k) \frac{d\bar{u}}{dt} \right] - (F_2 - k) \bar{u} = 0$$

through a continuous function of  $t$ , which, if  $k$  does not exceed certain limits, deviates indefinitely little throughout its whole trace from  $u$ , and may therefore be represented in the form

$$\bar{u} = u + (t, k),$$

where  $(t, k)$  becomes indefinitely small at the same time as  $k$  for all values of  $t$  that come into consideration.

The function  $\bar{u}$  will therefore vanish nowhere within the interval  $t_0 \dots t_1$ . In this manner a certain limit has also been established for  $k$ , which it cannot exceed; but if the condition is also

## CHAPTER XI.

THE NOTION OF A FIELD ABOUT THE CURVE WHICH OFFERS A  
MAXIMUM OR A MINIMUM VALUE OF THE INTEGRAL.

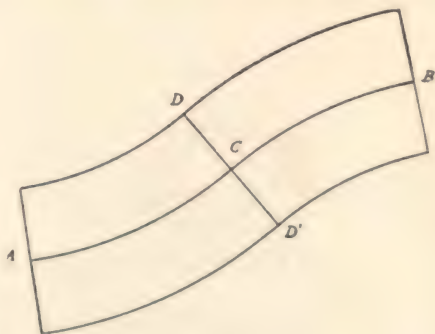
THE GEOMETRICAL MEANING OF THE  
CONJUGATE POINTS.

145. In Chapter IX we showed that the neighboring curves, which pass through a fixed point  $A$  and belong to the family of curves  $G = 0$ , intersect again in a point  $B$ , if  $B$  is the point conjugate to  $A$ . We shall now consider more fully this property of conjugate points.

We may first introduce the notion of a *field* about the curve which is to cause the integral to have a maximum or a minimum value.

We have assumed for all points belonging to that portion of the curve for which the integral in question is to be a maximum or a minimum that the function  $F(x, y, x', y')$  is regular in  $x, y, x'$  and  $y'$  and that  $F_1$  along this portion of curve is neither zero nor infinite. Exceptions to these assumptions are left for special investigation. From this, in connection with the necessary conditions already established, it is seen that the portion of curve can have no singular points (see Art. 95).

Such a portion of curve has therefore at every point only a single normal which cuts the curve, and the radii of curvature of all



points have a lower limit which is different from zero (Art. 80). Consequently, we may determine on both sides of the normal drawn through the point  $C$  of the curve  $AB$  two points  $D$  and  $D'$  in such a way that the normal within the interval  $DD'$  is not cut by any other normal to the curve in the neighborhood of the point  $C$ . Consider lengths similar to  $DD'$  drawn for all points along the curve; then the surface bounded by the points  $D$  and  $D'$ , which follow one another and which envelop completely the curve  $AB$ , has the property that within it no two normals drawn through two points of the curve that lie very close together intersect.

146. We represent the curve  $AB$ , which satisfies the differential equation  $G = 0$ , by the equations

$$1) \quad x = \phi(t, \alpha, \beta), \quad y = \psi(t, \alpha, \beta), \quad (t = t_0 \dots t_1),$$

and one of the neighboring curves, which also satisfies the differential equation  $G = 0$ , by the equations

$$2) \quad \bar{x} = \phi(t, \alpha + \alpha', \beta + \beta'), \quad \bar{y} = \psi(t, \alpha + \alpha', \beta + \beta').$$

Both curves are to pass through the same point  $A$ . If for the first curve there corresponds to the point  $A$  a definite value  $t_0$  of  $t$ , there will correspond to the same point for the second curve another value, say  $t_0 + \tau'$ .

The condition that the first curve shall cut the second curve is expressed by the two equations:

$$3) \quad \begin{cases} \phi(t_0 + \tau', \alpha + \alpha', \beta + \beta') - \phi(t_0, \alpha, \beta) = 0, \\ \psi(t_0 + \tau', \alpha + \alpha', \beta + \beta') - \psi(t_0, \alpha, \beta) = 0; \end{cases}$$

or, developed in powers of  $\tau'$ ,  $\alpha'$  and  $\beta'$ :

$$4) \quad \begin{cases} \phi'(t_0) \tau' + \phi_1(t_0) \alpha' + \phi_2(t_0) \beta' + (\tau', \alpha', \beta')_2 = 0, \\ \psi'(t_0) \tau' + \psi_1(t_0) \alpha' + \psi_2(t_0) \beta' + (\tau', \alpha', \beta')_2 = 0, \end{cases}$$

where  $(\tau', \alpha', \beta')_2$  denotes the terms of the second and higher powers of  $\tau'$ ,  $\alpha'$  and  $\beta'$ .

147. We may solve the equations 4) with respect to  $\tau'$ ,  $\alpha'$  and  $\beta'$  as follows. Suppose we have two equations

$$a x + b y + c z + (x, y, z)_2 = 0,$$

$$a' x + b' y + c' z + (x, y, z)_2 = 0,$$

where one of the three determinants  $a b' - a' b$ ,  $a c' - a' c$ ,  $b c' - b' c$  is different from zero. It follows,\* then, that we may express all values  $x, y, z$  which satisfy the two equations, and in which  $x, y, z$  do not exceed certain limits through three power-series of a single quantity.

We may choose for this quantity  $s = c_1 x + c_2 y + c_3 z$ , where  $c_1, c_2$  and  $c_3$  need satisfy the only condition:

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ c_1, & c_2, & c_3 \end{vmatrix} \neq 0.$$

For brevity, write (cf. Art. 126)

$$\psi_1(t_0) \phi_2(t_0) - \phi_1(t_0) \psi_2(t_0) = \theta_3(t_0);$$

then the three expressions corresponding in equations 4) to the determinants

$$a b' - a' b, a c' - a' c, b c' - b' c$$

are

$$5) \quad \begin{cases} \phi'(t_0) \psi_1(t_0) - \psi'(t_0) \phi_1(t_0) = -\theta_1(t_0), \\ \phi'(t_0) \psi_2(t_0) - \psi'(t_0) \phi_2(t_0) = -\theta_2(t_0), \\ \phi_1(t_0) \psi_2(t_0) - \psi_1(t_0) \phi_2(t_0) = -\theta_3(t_0), \end{cases}$$

of which  $\theta_1(t_0)$  and  $\theta_2(t_0)$  cannot both simultaneously vanish (Art. 127).

We may accordingly write

$$c_1 = 0, \quad c_2 \alpha' + c_3 \beta' = k_1,$$

\* See my lectures on the Theory of Maxima and Minima, etc., p. 102 and p. 21.

and further impose upon the constants  $c_2$  and  $c_3$  the condition

$$6) \quad \begin{vmatrix} \phi'(t_0), & \phi_1(t_0), & \phi_2(t_0) \\ \psi'(t_0), & \psi_1(t_0), & \psi_2(t_0) \\ 0, & c_2, & c_3 \end{vmatrix} = c_2\theta_2(t_0) - c_3\theta_1(t_0) = 1.$$

If we consider only the linear terms in equations 4), we have

$$\phi'(t_0) \tau' + \phi_1(t_0) \alpha' + \phi_2(t_0) \beta' = 0,$$

$$\psi'(t_0) \tau' + \psi_1(t_0) \alpha' + \psi_2(t_0) \beta' = 0,$$

$$c_2\alpha' + c_3\beta' = k_1.$$

From these equations we have as first approximations for  $\tau'$ ,  $\alpha'$  and  $\beta'$  the values

$$7) \quad \begin{cases} \tau' = -k_1 \theta_3(t_0), \\ \alpha' = +k_1 \theta_2(t_0), \\ \beta' = -k_1 \theta_1(t_0); \end{cases}$$

and therefore, finally,

$$7^a) \quad \begin{cases} \tau' = -k_1 \theta_3(t_0) + k_1^2 P_1(k_1, t_0), \\ \alpha' = +k_1 \theta_2(t_0) + k_1^2 P_2(k_1, t_0), \\ \beta' = -k_1 \theta_1(t_0) + k_1^2 P_3(k_1, t_0), \end{cases}$$

where  $P_1(k_1, t_0)$ ,  $P_2(k_1, t_0)$  and  $P_3(k_1, t_0)$  are power-series in  $k_1$  and  $t_0$ .

If we write these expressions in equations 2), or, what is the same thing, in

$$8) \quad \begin{cases} \bar{x} = \phi(t + \tau', \alpha + \alpha', \beta + \beta'), \\ \bar{y} = \psi(t + \tau', \alpha + \alpha', \beta + \beta'), \end{cases}$$

where, now,  $t$  may take values less than  $t_0$ , then we have

$$9) \quad \begin{cases} \bar{x} = x - k_1 [\phi'(t) \theta_3(t_0) - \phi_1(t) \theta_2(t_0) + \phi_2(t) \theta_1(t_0)] + k_1(t, k_1), \\ \bar{y} = y - k_1 [\psi'(t) \theta_3(t_0) - \psi_1(t) \theta_2(t_0) + \psi_2(t) \theta_1(t_0)] + k_1(t, k_1). \end{cases}$$

In these equations the symbol  $(t, k_1)$  is used to represent quantities which for every value of  $t$  become indefinitely small at the same time as  $k_1$ .

When  $k_1=0$ , the curve represented by equations 9) becomes the original curve, and we see that  $k_1$  can be taken so small that the two curves at corresponding points, that is, at points that belong to the same value of  $t$ , may come as near to each other as we wish. We shall show in the following Article that by this process we have derived all the curves that satisfy the differential equation  $G=0$ , which go through the point  $A$  and are neighboring the first curve.

148. Instead of the quantity  $k_1$  we may substitute a power-series in  $\tau', a', \beta'$  which is subjected only to the condition that if  $c_1, c_2, c_3$  are the coefficients of the linear terms in  $\tau', a', \beta'$ , the determinant

$$\begin{vmatrix} \phi'(t_0), & \phi_1(t_0), & \phi_2(t_0) \\ \psi'(t_0), & \psi_1(t_0), & \psi_2(t_0) \\ c_1, & c_2, & c_3 \end{vmatrix} \neq 0.$$

This condition is satisfied by the power-series which expresses the trigonometric tangent of the angle which the initial directions of the two curves at the point  $A=t_0$  include with each other.

For, denoting this tangent by  $k$ , we have

$$\begin{aligned} k &= \frac{\frac{dy_0}{dx_0} - \frac{d\bar{y}_0}{d\bar{x}_0}}{1 + \frac{dy_0}{dx_0} \frac{d\bar{y}_0}{d\bar{x}_0}} = \frac{\bar{x}_0' y_0' - \bar{y}_0' x_0'}{x_0' x_0' + y_0' y_0'} \\ &= \frac{\begin{vmatrix} \phi''(t_0), & \psi''(t_0) \\ \phi'(t_0), & \psi'(t_0) \end{vmatrix} \tau' + \begin{vmatrix} \phi_1'(t_0), & \psi_1'(t_0) \\ \phi'(t_0), & \psi'(t_0) \end{vmatrix} a' + \begin{vmatrix} \phi_2'(t_0), & \psi_2'(t_0) \\ \phi'(t_0), & \psi'(t_0) \end{vmatrix} \beta' + (\tau', a', \beta')_2}{\phi'^2(t_0) + \psi'^2(t_0) + (\tau', a', \beta')_1}. \end{aligned}$$

It is assumed that the curve is regular at the point  $A$ , so that the quantities  $\phi'(t_0)$  and  $\psi'(t_0)$  are not simultaneously zero, and consequently  $\phi'(t_0)^2 + \psi'(t_0)^2$  is different from zero.

Hence, the determinant of the equations 4) and 10) is

$$\frac{1}{\phi'^2(t_0) + \psi'^2(t_0)} \begin{vmatrix} \phi'(t_0), & \phi_1(t_0), & \phi_2(t_0), \\ \psi'(t_0), & \psi_1(t_0), & \psi_2(t_0), \\ \phi''(t_0), & \psi''(t_0) & \left| \begin{vmatrix} \phi_1'(t_0), & \psi_1'(t_0) \\ \phi_2'(t_0), & \psi_2'(t_0) \end{vmatrix} \right| \\ \phi'(t_0), & \psi'(t_0) & \left| \begin{vmatrix} \phi_1'(t_0), & \psi_1'(t_0) \\ \phi_2'(t_0), & \psi_2'(t_0) \end{vmatrix} \right| \end{vmatrix}.$$

Multiply the first horizontal row by  $\psi''(t_0)$ , the second by  $-\phi''(t_0)$ , and add them both to the third row, which then becomes

$$o, \quad \begin{vmatrix} \phi_1'(t_0), & \psi_1'(t_0) \\ \phi'(t_0), & \psi'(t_0) \end{vmatrix} + \begin{vmatrix} \phi_1(t_0), & \psi_1(t_0) \\ \phi''(t_0), & \psi''(t_0) \end{vmatrix},$$

$$\begin{vmatrix} \phi_2'(t_0), & \psi_2'(t_0) \\ \phi'(t_0), & \psi'(t_0) \end{vmatrix} + \begin{vmatrix} \phi_2(t_0), & \psi_2(t_0) \\ \phi''(t_0), & \psi''(t_0) \end{vmatrix};$$

or, what is the same thing,

$$o, \quad \theta_1'(t_0), \quad \theta_2'(t_0).$$

Hence, the above determinant is

$$10) \quad \frac{1}{\phi'^2(t_0) + \psi'^2(t_0)} [\theta_1'(t_0) \theta_2(t_0) - \theta_2'(t_0) \theta_1(t_0)]$$

$$= \frac{1}{\phi_1^2(t_0) + \psi^2(t_0)} \frac{C}{F_1(t_0)} \text{ (see Art. 129),}$$

an expression which (*loc. cit.*) is different from zero.

We may accordingly write  $k$  in the place of  $k_1$ , and find in the same way as above:

$$11) \quad \begin{cases} \bar{x} = x + k f_1(t, k) \\ \bar{y} = y + k f_2(t, k) \end{cases}$$

149. In Art. 89 the form of the solution of the differential equation  $G=0$  was given. *It follows that a curve which satisfies the equation  $G=0$  is completely determined as soon as its initial point and the direction of the tangent at this point are known.*

Let  $a, b$  be the coordinates of  $A$  and  $\lambda$  (see Fig. of Art. 87) the angle which the initial direction makes with the  $X$ -axis; further take instead of the coordinates  $x, y$  a new system of coordinates  $t, v$  with a new origin at  $A$  in such a way that

$$12) \quad \begin{cases} t = (x-a) \cos \lambda - (y-b) \sin \lambda, \\ v = (x-a) \sin \lambda + (y-b) \cos \lambda; \end{cases}$$

or

$$13) \quad \begin{cases} x = a + t \cos \lambda + v \sin \lambda, \\ y = b - t \sin \lambda + v \cos \lambda. \end{cases}$$

Now if we choose  $t$  as the independent variable, then is

$$\begin{aligned} x' &= \cos \lambda + \frac{dv}{dt} \sin \lambda, & \frac{dx'}{dt} &= \frac{d^2v}{dt^2} \sin \lambda, \\ y' &= -\sin \lambda + \frac{dv}{dt} \cos \lambda, & \frac{dy'}{dt} &= \frac{d^2v}{dt^2} \cos \lambda; \end{aligned}$$

and consequently

$$x' \frac{dy'}{dt} - y' \frac{dx'}{dt} = \frac{d^2v}{dt^2}.$$

The differential equation  $G=0$ , *i. e.*,

$$F_1 \left( x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) + H(x, y, x', y') = 0,$$

becomes then

$$14) \quad 0 = \frac{d^2v}{dt^2} F_1 \left( a + t \cos \lambda + v \sin \lambda, b - t \sin \lambda + v \cos \lambda, \cos \lambda + \frac{dv}{dt} \sin \lambda, -\sin \lambda + \frac{dv}{dt} \cos \lambda \right) + H \left( t, v, \frac{dv}{dt} \right) \quad (\text{Art. 94}).$$

Following the method of integration given in Chapter VI, we solve the above equation in such a way that when  $t=0$ , both  $v=0$  and  $\frac{dv}{dt}=0$ , the  $v$ -axis being the direction of the tangent at the point  $A$ .

Hence, if  $F_1(a, b, \cos \lambda, -\sin \lambda)$  has a finite value different from zero, and if  $H\left(t, v, \frac{dv}{dt}\right)$  does not become infinitely large at the point  $A$ , as we have assumed was the case, since  $F_1$  together with its derivatives, of which  $H$  consists, is a regular function of its arguments, it follows that there is only one power-series of  $t$  that satisfies the differential equation, and which with its first derivative vanishes for  $t=0$ .

This power-series has the form

$$v=t^2 P(t).$$

Writing this value of  $v$  in the equations 13), they become

$$15) \quad \begin{cases} x=a+t \cos \lambda + A_1 t^2 + \dots, \\ y=b-t \sin \lambda + B_1 t^2 + \dots, \end{cases}$$

where the constants  $A_1, B_1, \dots$  are definitely determined.

Thus the equations 15) completely determine the curve which satisfies the differential equation  $G=0$ , where  $a, b$  are the coordinates of its initial point and  $\lambda$  the angle which its initial direction makes with the  $X$ -axis.

From this it follows at once that through equations 11) we have all the neighboring curves of the original curve which pass through the same initial point and satisfy the differential equation  $G=0$ .

150. We may therefore give  $k$  an upper limit in such a way that all curves belonging to a value of  $k$  below this limit and satisfying the differential equation  $G=0$  lie completely in the surface which envelops the original curve.

This makes it possible to bring about a one-valued relation between both curves in such a way that, corresponding to every point of the original curve, we may determine the point of the neighboring curve at which this curve is cut by the normal at a point on the first curve.

Let  $x, y$  be the coordinates of a point  $P$  on the original curve, and  $x+\xi, y+\eta$  the coordinates of the corresponding point on the neighboring curve.

If  $P'$  is the point corresponding to  $P$ , its coordinates are

$$x + \xi = \phi(t + \tau, a + a', \beta + \beta'),$$

$$y + \eta = \psi(t + \tau, a + a', \beta + \beta');$$

and besides, since  $(X - x)x' + (Y - y)y' = 0$  is the equation of the normal, and  $X = x + \xi$ ,  $Y = y + \eta$  is a point on it, we have

$$x'\xi + y'\eta = 0.$$

Hence,  $\xi$ ,  $\eta$  and  $\tau$  are to be determined from the equations

$$16) \quad \begin{cases} \xi = \phi'(t) \tau + \phi_1(t) a' + \phi_2(t) \beta' + \dots, \\ \eta = \psi'(t) \tau + \psi_1(t) \beta' + \psi_2(t) \beta' + \dots, \\ 0 = \phi'(t) \xi + \psi'(t) \eta. \end{cases}$$

The last of these equations combined with the first and second gives

$$17) \quad 0 = [\phi'^2(t) + \psi'^2(t)]\tau + k f(t) + (t, k),$$

when for  $a'$ ,  $\beta'$  we have written from 7<sup>a</sup>) their power-series in  $k$ .

Since the portion of curve  $AB$  has no singularity, and consequently  $\phi'^2(t) + \psi'^2(t)$  nowhere vanishes, we may from equation 17) express  $\tau$  and therefore also  $\xi$  and  $\eta$  as power-series in  $k$ . If, then, we limit ourselves to curves with which  $k$  remains within a certain limit, we may always determine the point where such a curve is cut by a normal of the original curve.

151. We ask if it is possible for the second curve to intersect the first curve. For this to be the case the length  $PP'$  must be zero; that is,  $\xi$ ,  $\eta$  must for some value of  $t$  be equal to zero. Hence we have so to choose the quantities  $t, \tau, \tau', a', \beta'$  that the equations 4) and 16), when in 16)  $\xi$  and  $\eta$  are put equal to zero, are satisfied.

The terms of like dimension in 4) and 16) are homogeneous functions of  $\tau', a', \beta'$  and of  $\tau, a' \beta'$  respectively; these equations may be written:

$$\{\phi'(t_0) + v\}\tau' + \{\phi_1(t_0) + p\}a' + \{\phi_2(t_0) + q\}\beta' = 0,$$

$$\{\psi'(t_0) + v_1\}\tau' + \{\psi_1(t_0) + p_1\}a' + \{\psi_2(t_0) + q_1\}\beta' = 0,$$

$$\{\phi'(t) + v_2\}\tau + \{\phi_1(t) + p_2\}a' + \{\phi_2(t) + q_2\}\beta' = 0,$$

$$\{\psi'(t) + v_3\}\tau + \{\psi_1(t) + p_3\}a' + \{\psi_2(t) + q_3\}\beta' = 0,$$

where  $v, p, q, v_1, p_1, q_1$  represent functions of  $\tau', a', \beta'$ ; and  $v_2, p_2, q_2, v_3, p_3, q_3$  are functions of  $\tau, a', \beta'$ , which with these functions and therefore also with  $k$ , become infinitely small.

The first two of these equations express that the two neighboring curves pass through the initial point  $A$ , and the last two that they are to go through another point.

In order that these four equations exist simultaneously, their determinant must vanish. This determinant, when in it we make  $k=0$ , is :

$$\begin{vmatrix} \phi'(t_0), & 0, & \phi_1(t_0), & \psi_2(t_0) \\ \psi'(t_0), & 0, & \psi_1(t_0), & \psi_2(t_0) \\ 0, & \phi'(t), & \phi_1(t), & \phi_2(t) \\ 0, & \psi'(t), & \psi_1(t), & \psi_2(t) \end{vmatrix},$$

and this is nothing other than the function  $-\Theta(t, t_0)$ . Hence the determinant of the above system of equations may be brought to the form  $-\Theta(t, t_0) - k(t, t_0, k)$ ; and, as this determinant is to vanish, we must have

$$18) \quad \Theta(t, t_0) + k(t, t_0, k) = 0.$$

If, now,  $C$  is a point of the original curve for which  $t=t'$  and which is not conjugate to  $A$ , then  $\Theta(t', t_0)$  is different from zero, and we may therefore fix a limit for  $k$  so that for all values of  $k$  under this limit the expression  $\Theta(t', t_0) + k(t', t_0, k)$  is different from zero; that is, none of the curves which lie very near the original curve can cut this curve at the point  $t'$  or in the neighborhood of it, since we can always find a limit  $h$  of such a nature that for every value of  $t$  within the interval  $t' - h \dots t' + h$  the expression is different from zero. And, reciprocally, every curve that lies

very near the original curve will cut this curve in the neighborhood of  $C$ ; as soon as there is a point  $C$  in the interval  $AB$  which is conjugate to  $A$ . For one can then always find for  $k$  a value sufficiently small that, with very small values of  $h$ , the sign of  $\Theta(t' - h, t_0) + k(t' - h, t_0, k)$  is the same as the sign of  $\Theta(t' - h, t_0)$ , and the sign of  $\Theta(t' + h, t_0) + k(t' + h, t_0, k)$  is the same as that of  $\Theta(t' + h, t_0)$ . But when the function  $\Theta(t, t_0)$  passes through the value zero it changes its sign, as is seen in the following Article. Hence, it follows, as  $\Theta(t', t_0)$  is to be zero, that the expression  $\Theta(t, t_0) + k(t, t_0, k)$  must vanish once within the interval  $t' - h \dots t' + h$ ; or, in other words: *If, in the interval  $AB$  of the original curve, there is a point  $t=t'$  conjugate to the initial point, then all the curves which lie very close to the first curve, which satisfy the differential equation  $G=0$  and which have the same initial point  $A$ , will cut again the first curve in the neighborhood of the point  $t'$ . Consequently the conjugate point is nothing other than the limiting position which the points of intersection of a neighboring curve with the original curve approach, if we make smaller and smaller the angle which the initial directions of the two curves make with each other.*

If there is no such limiting position within the interval  $AB$ , then there is no conjugate point within this interval.

152. It remains yet to show that the point  $A$  cannot itself be this limiting position; that is, of all the neighboring curves there cannot be one which cuts the original curve as close as we wish to  $A$ . Analytically this case may be expressed in the following manner: If at the point  $t$  the original curve is cut by a neighboring curve, we have the equation

$$\begin{vmatrix} 0, & \phi'(t_0) + v, & \phi_1(t_0) + p, & \phi_2(t_0) + q \\ 0, & \psi'(t_0) + v_1, & \psi_1(t_0) + p_1, & \psi_2(t_0) + q_1 \\ \phi'(t) + v_2, & 0, & \phi_1(t) + p_2, & \phi_2(t) + q_2 \\ \psi'(t) + v_3, & 0, & \psi_1(t) + p_3, & \psi_2(t) + q_3 \end{vmatrix} = 0,$$

a determinant which becomes  $\Theta(t, t_0) = 0$ , when  $k = 0$ . If for  $\tau, \tau', \alpha', \beta'$  expressed as power-series in  $k$ , their values be substituted in the determinant, it becomes an equation in  $t$  and  $k$ . Further, since

$\phi'(t), \phi_1(t), \dots, \psi_2(t)$  are power-series in  $t$  which are regular functions in the neighborhood of  $t_0$ , the determinant may be developed in a power-series in  $t-t_0$  and  $k$ , which converges for sufficiently small values of  $t-t_0$  and  $k$ . If in the neighborhood of the original curve there exist curves which cut this curve as near as we wish to  $A$ , then, after sufficiently small limits have been given to  $t-t_0$  and  $k$ , it is possible to find values for these quantities within the given limits for which the equation is satisfied.

If we write  $t=t_0$ , the quantities  $v, p, q$  are respectively equal to  $v_2, p_2, q_2$  and the quantities  $v_1, p_1, q_1$  to  $v_3, p_3, q_3$ .

When this is the case, the determinant has the form

$$\begin{vmatrix} o, & a, & b, & c \\ o, & a_1, & b_1, & c_1 \\ a, & o, & b, & c \\ a_1, & o, & b_1, & c_1 \end{vmatrix},$$

which is identically zero. Therefore the power-series in  $t-t_0$  and  $k$  will vanish for  $t=t_0$ , whatever be the value of  $k$ ; and consequently this series is divisible by  $t-t_0$ .

The determinant, then, when divided by  $t-t_0$  is for the value  $t=t_0$ :

$$19) \quad \left[ \frac{d\Theta(t, t_0)}{dt} \right]_{t=t_0} + (t-t_0, k)_{t=t_0} = o.$$

We saw in Arts. 128 and 129 that

$$\Theta(t, t_0) = \theta_1(t_0) \theta_2(t) - \theta_2(t_0) \theta_1(t),$$

and that

$$\theta_1(t) \theta'_2(t) - \theta_2(t) \theta'_1(t) = \frac{C}{F_1(t)}.$$

If  $t'$  is a conjugate point to  $t_0$ , so that

$$\Theta(t', t_0) = \theta_1(t_0) \theta_2(t') - \theta_2(t_0) \theta_1(t') = o,$$

it follows that

$$\theta_1(t') = \lambda \theta_1(t_0),$$

$$\theta_2(t') = \lambda \theta_2(t_0),$$

where  $\lambda$  is a constant different from zero.

We further have, since

$$\frac{d}{dt} \Theta(t, t_0) = \theta_1(t_0) \theta_2'(t) - \theta_2(t_0) \theta_1'(t),$$

the relation

$$\left[ \frac{d}{dt} \Theta(t, t_0) \right]_{t=t'} = \frac{1}{\lambda} \frac{C}{F_1(t')},$$

which is different from zero.

It is thus seen that the derivative of  $\Theta(t, t_0)$  does not vanish on the positions at which the function itself vanishes.

At the same time it is shown that the equation 19) is not satisfied, so long as  $k$  and  $t - t_0$  remain within finite limits; and consequently a neighboring curve cannot intersect the original curve a second time indefinitely near the initial point through which both curves pass.

As there is a great range of choice regarding the variable  $t$ , and as the constants  $\alpha$  and  $\beta$  may be chosen in many ways, it is possible to give many forms to the function  $\Theta$ . To be strictly rigorous, it would yet remain to prove that the solution of the equation  $\Theta(t, t_0)$  leads always to the same conjugate point, whatever be the form of  $\Theta$ ; the geometrical significance of these points, however, make such a proof superfluous.

## CHAPTER XII.

A FOURTH AND FINAL CONDITION FOR THE EXISTENCE OF A  
MAXIMUM OR A MINIMUM, AND A PROOF THAT THE  
CONDITIONS WHICH HAVE BEEN GIVEN  
ARE SUFFICIENT.

153. In the preceding Chapter we considered the family of curves that have the same initial point  $A$  and satisfy the differential equation  $G=0$ . These deviate very little from one another in their initial direction. We saw that the curves again intersect only in the neighborhood of points that are the conjugates of  $A$ , the conjugate point along any curve being the limiting position of the point of intersection of this curve and a neighboring curve when the angle between their initial directions becomes infinitesimally small. All points that lie on these curves before the points that are conjugate to  $A$  form a connected portion of surface; that is, if  $P_1$  is a point belonging to this collectivity of points, a boundary may be described about  $P_1$  so that all points within this boundary also belong to the collectivity of points.

For, let

$$x = \phi(t, a, \beta), \quad y = \psi(t, a, \beta)$$

be the equations of a given curve which satisfies  $G=0$ , and let the coordinates of a point on this curve be

$$x_1 = \phi(t_1, a, \beta), \quad y_1 = \psi(t_1, a, \beta).$$

Further, let  $x_1 + \xi, y_1 + \eta$  be the coordinates of another point  $P_2$  that lies in the neighborhood of  $P_1$ , so that  $\xi, \eta$  are quantities arbitrarily small.

We may then (Art. 151) draw a curve between  $A$  and  $P_2$  which satisfies the differential equation  $G=0$ , if we can determine four quantities  $\tau, \tau', \alpha', \beta'$  as power-series in  $\xi, \eta$  in such a way that the following equations are true:

$$0 = \phi'(t_0) \tau' + \phi_1(t_0) \alpha' + \phi_2(t_0) \beta' + (\tau', \alpha', \beta')_2,$$

$$0 = \psi'(t_0) \tau' + \psi_1(t_0) \alpha' + \psi_2(t_0) \beta' + (\tau', \alpha', \beta')_2,$$

$$\xi = \phi'(t_1) \tau + \phi_1(t_1) \alpha' + \phi_2(t_1) \beta' + (\tau, \alpha', \beta')_2,$$

$$\eta = \psi'(t_1) \tau + \psi_1(t_1) \alpha' + \psi_2(t_1) \beta' + (\tau, \alpha', \beta')_2.$$

Since the determinant of these equations (Art. 151) is  $-\Theta(t_1, t_0)$  and is different from zero, the point  $t_1$  not being conjugate to  $t_0$ , it follows that the quantities  $\tau, \tau', \alpha', \beta'$  may be developed in power-series in  $\xi, \eta$  which are convergent for small values of these quantities.

Consequently a curve may be drawn through  $A$  and  $P_2$  which satisfies the differential equation  $G=0$ , and this curve will be neighboring the first curve and will deviate as little as we wish in direction from its initial direction, if  $\xi, \eta$ , and consequently also  $\tau, \tau', \alpha', \beta'$ , are sufficiently small.

If we form the determinant for the curve  $AP_2$ , which, when put equal to zero, is the equation for the determination of the point conjugate to  $A$ , it is seen that this determinant also may be developed as a power-series in  $\xi, \eta$ , which becomes  $-\Theta(t_1, t_0)$  when  $\xi=\eta=0$ . The function  $\Theta(t_1, t_0)$  is different from zero when sufficiently small values are ascribed to  $\xi, \eta$ . Consequently within the interval  $AP_2$  there is present no point which is conjugate to  $A$ .

*We may therefore envelop the interval situated between two conjugate points of the original curve by a narrow surface area, which is of such a nature that a curve, and only one, may be drawn from the point  $A$  to any point within it, which satisfies the differential equation  $G=0$ , is neighboring the first curve and deviates in its initial direction only a little from it.*

154. Let a portion of curve  $P_0P_1$ , satisfying the differential equation  $G=0$ , be given, which is of such a nature that for no

point on it  $F_1=0$  or  $\infty$ , and suppose that the point conjugate to  $P_0$  does not lie before  $P_1$ . Between  $P_0$  and  $P_1$  take an arbitrary point



$P_2$  and draw through  $P_2$  a regular curve.\* On this curve we choose a point  $P_3$  so close to  $P_2$  that a curve may be drawn through  $P_0$  and  $P_3$  which satisfies the differential equation  $G=0$ , and which lies entirely within the strip of surface defined above.

Let us consider the change in the integral when we take it over  $P_0P_3+P_3P_2$  instead of over  $P_0P_2$ . We may denote an integral taken over a curve that satisfies the differential equation by  $I$ , and one over an arbitrary curve by  $\bar{I}$ , and we may denote the direction of integration by added indices. We have therefore to compute the expression

$$\Delta I = I_{03} + \bar{I}_{32} - I_{02},$$

or

$$\Delta I = \left( \int_{P_0 P_3} F dt - \int_{P_0 P_2} F dt \right) + \int_{P_3 P_2} F dt,$$

an expression which (Art. 79)

$$= \epsilon \left\{ \int_{P_0 P_2} G w ds + \left[ \xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'} \right]^{t_2} \right\} + (\epsilon^2) + \int_{P_3 P_2} F dt,$$

where  $\xi, \eta$  are measured in the direction from  $P_2$  to  $P_3$ .

At the point  $P_2$  and along the curve  $P_3P_2$  in the direction  $P_3P_2$  we have

$$\epsilon \xi = -\bar{x}_2' d\bar{t} + (d\bar{t})^2,$$

$$\epsilon \eta = -\bar{y}_2' d\bar{t} + (d\bar{t})^2,$$

$d\bar{t}$  denoting that this differential is taken with respect to the curve  $P_3P_2$ .

If we consider the arguments in  $F$  expressed as functions of  $\bar{t}$  along the curve  $P_3P_2$ , it follows that

$$\int_{P_3 P_2} F dt = F(x_2, y_2, \bar{x}_2', \bar{y}_2') d\bar{t} + (d\bar{t})^2.$$

\*The shaded curves do not satisfy the differential equation  $G=0$ .

Hence at the point  $P_2$ , which is an arbitrary point of the curve  $P_0P_1$ , we have

$$\Delta I = \left\{ F(x_2, y_2, \bar{x}_2', \bar{y}_2') - \left[ \bar{x}_2' \frac{\partial}{\partial x_2'} F(x_2, y_2, x_2', y_2') + \bar{y}_2' \frac{\partial}{\partial y_2'} F(x_2, y_2, x_2', y_2') \right] \right\} d\bar{t} + (d\bar{t})^2. \quad (a)$$

The function  $F$  is homogeneous of the first order (Art. 68) with respect to its third and fourth arguments, so that (see Art. 72)

$$F(x_2, y_2, \bar{x}_2', \bar{y}_2') = \bar{x}_2' F^{(1)}(x_2, y_2, \bar{x}_2', \bar{y}_2') + \bar{y}_2' F^{(2)}(x_2, y_2, \bar{x}_2', \bar{y}_2').$$

We define by  $\mathfrak{E}(x, y, x', y', \bar{x}', \bar{y}')$  the expression

$$1) \quad \mathfrak{E}(x, y, x', y', \bar{x}', \bar{y}') = \bar{x}' \{ F^{(1)}(x, y, \bar{x}', \bar{y}') - F^{(1)}(x, y, x', y') \} + \bar{y}' \{ F^{(2)}(x, y, \bar{x}', \bar{y}') - F^{(2)}(x, y, x', y') \}.$$

Hence at the point  $P_2$  it follows that

$$\Delta I = \mathfrak{E}(x, y, x', y', \bar{x}', \bar{y}') d\bar{t} + (d\bar{t})^2,$$

when in the function  $\mathfrak{E}$  we have substituted for the arguments those values that belong to the point  $P_2$ . The direction-cosines of the curve  $P_0P_2$  at  $P_2$  are denoted by

$$p_2 = \frac{x_2'}{\sqrt{x_2'^2 + y_2'^2}} \quad \text{and} \quad q_2 = \frac{y_2'}{\sqrt{x_2'^2 + y_2'^2}},$$

and those of the curve  $P_3P_2$  at  $P_2$  by  $\bar{p}_2$  and  $\bar{q}_2$ . It is evident from a consideration of the right-hand side of the formula defining  $\mathfrak{E}$  above (and cf. Art. 68) that

$$\frac{\mathfrak{E}(x, y, x', y', \bar{x}', \bar{y}')}{\sqrt{\bar{x}'^2 + \bar{y}'^2}} = \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}).$$

155. If further we denote by  $\sigma$  the differential of arc  $P_3P_2$ , we have finally

$$2) \quad \Delta I = \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q})\sigma + (\sigma)^2.$$

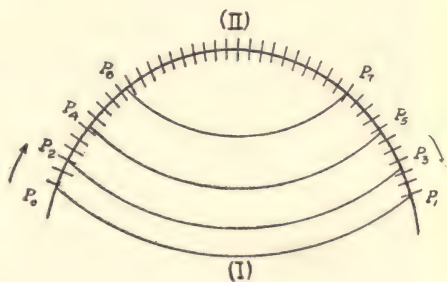
Accordingly, if we take  $\sigma$  sufficiently small; that is, if we choose the point  $P_3$  very close to  $P_2$ , then we may always bring it about that the change in the integral has the same sign as that of the function  $\mathfrak{E}$ .

The point  $P_2$  was an arbitrary point on the curve  $P_0P_1$ , and  $P_2P_3$  also represented an arbitrary direction.

It follows that if for any point  $P_2$  and for any direction at  $P_2$  the function  $\mathfrak{E}$  were *negative*, and for any other point and direction *positive*, then the given curve could vary in such a manner that the change in the integral is at one time *positive* and at another time *negative*. We have, therefore, the following theorem:

*If the integral taken over the curve  $P_0P_1$  which satisfies the differential equation  $G=0$  is to be a maximum or a minimum, then the function  $\mathfrak{E}$  must have the same sign for every point of the curve, and at every point of the curve for any direction, and this sign must be negative for a maximum and positive for a minimum.*

156. That the above condition is sufficient to assure the existence of a maximum or a minimum may be shown as follows: Let  $P_0(\text{I})P_1$  be a curve which satisfies the *four* conditions already established (and recapitulated in Art. 174), and let  $P_0(\text{II})P_1$  be any arbitrary curve that lies in the field about the curve  $P_0(\text{I})P_1$ . It is subject only to the condition that it must be a regular curve and lie wholly in the given field.



By varying the parameters  $a$  and  $\beta$  we can construct a system of curves as near as we like to one another, all satisfying the differential equation  $G=0$ . These curves cut the curve  $P_0(\text{II})P_1$  in two (or perhaps more) points. They do not cut the curve  $P_0(\text{I})P_1$  or intersect among themselves within the field in question. The function  $\mathfrak{E}$  must have the same sign along each of these curves as it has along the curve  $P_0(\text{I})P_1$ . For, take an arbitrary point  $P$  on

any of these curves. Then on the curve  $P_0(\text{I})P_1$  there is a point for which the quantities  $x, y, p, q$  differ only a little from the quantities that belong to the point  $P$ , and consequently  $\mathfrak{E}$  has the same sign for both points.

Consider now the variation in our integrals as we pass from  $P_0(\text{I})P_1$  to  $P_0P_2P_3P_1$  and from  $P_0P_2P_3P_1$  to  $P_0P_4P_5P_1$ , etc. As we saw in the preceding article, the variation caused by passing from  $P_0(\text{I})P_1$  to  $P_0P_2P_3P_1$

$$\begin{aligned} &= \int_{t_0}^{t_2} F dt + \epsilon \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t_2}^{t_1} + \int_{t_3}^{t_1} F dt \\ &= \int_{t_0}^{t_2} F dt - \left[ \bar{p} \frac{\partial F}{\partial x'} + \bar{q} \frac{\partial F}{\partial y'} \right]_{t_2}^{t_1} \sigma + \int_{t_3}^{t_1} F dt - \left[ \bar{p} \frac{\partial F}{\partial x'} + \bar{q} \frac{\partial F}{\partial y'} \right]_{t_3}^{t_1} \sigma, \end{aligned}$$

$\bar{p}, \bar{q}$  being the direction-cosines of the tangent to the curve  $P_0(\text{II})P_1$  at the points  $P_2$  and  $P_3$ , which, we notice, have opposite signs at these points.

If we denote the integration along the curves by the curves themselves, it is seen at once that the variation in these integrals may be expressed by

$$P_0P_2P_3P_1 - P_0(\text{I})P_1 = [\mathfrak{E}]^{t_0} \sigma + [\mathfrak{E}]^{t_1} \sigma + (\sigma)^2,$$

where the first  $\sigma$  is the length from  $P_0$  to  $P_2$  and the second from  $P_3$  to  $P_1$ .

Similarly the differences in the integrals along

$$P_0P_4P_5P_1 - P_0P_2P_3P_1 = [\mathfrak{E}]^{t_2} \sigma + [\mathfrak{E}]^{t_3} \sigma + (\sigma)^2,$$

$$P_0P_6P_7P_1 - P_0P_4P_5P_1 = [\mathfrak{E}]^{t_4} \sigma + [\mathfrak{E}]^{t_5} \sigma + (\sigma)^2,$$

$$\dots \dots \dots$$

$$P_0P_{2\nu}P_{2\nu+1}P_1 - P_0P_{2\nu-2}P_{2\nu-1}P_1 = [\mathfrak{E}]^{t_{2\nu-2}} \sigma + [\mathfrak{E}]^{t_{2\nu-1}} \sigma + (\sigma)^2,$$

$$P_0(\text{II})P_1 - P_0P_{2\nu}P_{2\nu+1}P_1 = [\mathfrak{E}]^{t_{2\nu}} \sigma + [\mathfrak{E}]^{t_{2\nu+1}} \sigma + (\sigma)^2.$$

Adding these results together, we have the difference in the integrals along

$$P_0(\text{II})P_1 - P_0(\text{I})P_1 = \int_{P_0(\text{II})P_1} \mathfrak{E} \sigma + (\sigma)^2,$$

$\sigma$  being a differential of arc along the curve  $P_0(\text{II})P_1$ . This is a verification of the theorem stated at the end of the last Article.

We also see that, if we had not assured ourselves that none of the intermediary curves intersect, the signs of the  $\sigma$ 's would not all have been alike, and consequently the sum total of all these  $\sigma$ 's would not have constituted the curve  $P_0(\text{II})P_1$ .

### 157. *Another form of the function $\mathfrak{E}$ .*

We have seen in the Integral Calculus that

$$\begin{aligned} f(p_1, q_1) - f(p_0, q_0) &= \int_{p_0, q_0}^{p_1, q_1} df(p, q) \\ &= \int_{p_0, q_0}^{p_1, q_1} \left( \frac{\partial f(p, q)}{\partial p} dp + \frac{\partial f(p, q)}{\partial q} dq \right) \\ &= \int_{k=0}^{k=1} \left( f^{(1)}[p_0 + k(p_1 - p_0), q_0 + k(q_1 - q_0)](p_1 - p_0) \right. \\ &\quad \left. + f^{(2)}[p_0 + k(p_1 - p_0), q_0 + k(q_1 - q_0)](q_1 - q_0) \right) dk. \end{aligned}$$

Hence, if we write

$$\begin{cases} p_k = p + k(\bar{p} - p) = (1-k)p + k\bar{p}, \\ q_k = q + k(\bar{q} - q) = (1-k)q + k\bar{q}, \end{cases}$$

it is seen that

$$F^{(1)}(x, y, \bar{p}, \bar{q}) - F^{(1)}(x, y, p, q) = \int_{k=0}^{k=1} \left( F^{(11)}(x, y, p_k, q_k) (\bar{p} - p) \right. \\ \left. + F^{(12)}(x, y, p_k, q_k) (\bar{q} - q) \right) dk,$$

$$F^{(2)}(x, y, \bar{p}, \bar{q}) - F^{(2)}(x, y, p, q) = \int_{k=0}^{k=1} \left( F^{(21)}(x, y, p_k, q_k) (\bar{p} - p) \right. \\ \left. + F^{(22)}(x, y, p_k, q_k) (\bar{q} - q) \right) dk.$$

Note that (see Art. 73)

$$F^{(11)} = q_k^2 F_1, \quad F^{(12)} = -p_k q_k F_1, \quad F^{(22)} = p_k^2 F_1,$$

and further that  $F^{(12)} = F^{(21)}$ .

By substituting these values in the above expressions, and in turn the resulting quantities in the expression for  $\mathfrak{E}$ , we have

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = \bar{p} [F^{(1)}(x, y, \bar{p}, \bar{q}) - F^{(1)}(x, y, p, q)] \\ + \bar{q} [F^{(2)}(x, y, \bar{p}, \bar{q}) - F^{(2)}(x, y, p, q)] \\ = \int_{k=0}^{k=1} F_1(x, y, p_k, q_k) \left( [q_k(\bar{p} - p) - p_k(\bar{q} - q)] q_k \bar{p} \right. \\ \left. + [-q_k(\bar{p} - p) + p_k(\bar{q} - q)] p_k \bar{q} \right) dk.$$

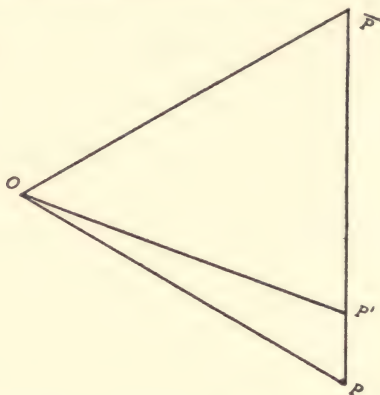
The expression in the square brackets is

$$(q_k \bar{p} - p_k \bar{q}) [q_k(\bar{p} - p) - p_k(\bar{q} - q)] = (1 - k) (q \bar{p} - p \bar{q})^2,$$

and consequently

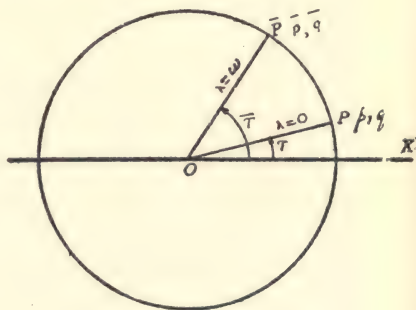
$$3) \quad \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = (q \bar{p} - p \bar{q})^2 \int_{k=0}^{k=1} F_1(x, y, p_k, q_k) (1 - k) dk.$$

This expression for  $\mathfrak{E}$  in the form of a definite integral is defective, in that it has a meaning only when  $F_1$  remains finite for all values of  $p_k$  and  $q_k$ , as  $k$  varies between 0 and 1. For example, if  $k = \frac{1}{2}$ , then  $p_{\frac{1}{2}} = p + \frac{1}{2}(\bar{p} - p) = \frac{1}{2}(p + \bar{p})$ , and if  $\bar{p} = -p$ , then  $p_{\frac{1}{2}} = 0$ ; in the same way for  $k = \frac{1}{2}$  and  $\bar{q} = -q$ , then also  $q_{\frac{1}{2}} = 0$ . These two arguments being zero,  $F_1$  becomes infinite (cf. Art. 73). Further, if the two directions  $p, q$  and  $\bar{p}, \bar{q}$  coincide, then  $\mathfrak{E}$  becomes zero of the second order.



If  $OP$  and  $O\bar{P}$  are vectors of unit length with components  $p, q$  and  $\bar{p}, \bar{q}$ , then the components of  $OP'$ , when  $P'$  travels along the line  $P\bar{P}$ , are  $p_k, q_k$ ,  $k$  varying between 0 and 1.

158. Another form was given by Weierstrass to the expression  $\mathfrak{E}$ , in which he avoided the defect mentioned above, by integrating along the arc of a circle instead of along the straight line  $P\bar{P}$ . If we integrate along the arc of a circle of unit radius from the point  $P$  to the point  $\bar{P}$  we obtain an expression for  $\mathfrak{E}$  which is universally true.



We have as before, if  $POX = \tau$ ,  $\bar{P}OX = \bar{\tau}$ ,  $\omega = \bar{\tau} - \tau \pmod{2\pi}$  and  $-\pi \leq \omega \leq +\pi$ ,

$$\begin{aligned} \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = & \bar{p}[F^{(1)}(x, y, \bar{p}, \bar{q}) - F^{(1)}(x, y, p, q)] \\ & + \bar{q}[F^{(2)}(x, y, \bar{p}, \bar{q}) - F^{(2)}(x, y, p, q)] \end{aligned}$$

$$\begin{aligned}
 &= \cos \bar{\tau} [F^{(1)}(x, y, \cos \bar{\tau}, \sin \bar{\tau}) - F^{(1)}(x, y, \cos \tau, \sin \tau)] \\
 &+ \sin \bar{\tau} [F^{(2)}(x, y, \cos \bar{\tau}, \sin \bar{\tau}) - F^{(2)}(x, y, \cos \tau, \sin \tau)] \\
 &= \cos \bar{\tau} \int_{\lambda=0}^{\lambda=\omega} d_{\lambda} F^{(1)}[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] \\
 &+ \sin \bar{\tau} \int_{\lambda=0}^{\lambda=\omega} d_{\lambda} F^{(2)}[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)].
 \end{aligned}$$

But, if  $F^{(1)}$  denotes the derivative of  $F$  with respect to its third argument, etc.,

$$\begin{aligned}
 d_{\lambda} F^{(1)}[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] \\
 &= [-F_{\cos, \cos}^{(11)} \sin(\tau + \lambda) + F_{\cos, \sin}^{(12)} \cos(\tau + \lambda)] d\lambda \\
 &= [-\sin^3(\tau + \lambda) - \sin(\tau + \lambda) \cos^2(\tau + \lambda)] F_1 d\lambda \\
 &= -\sin(\tau + \lambda) F_1[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] d\lambda;
 \end{aligned}$$

similarly,

$$\begin{aligned}
 d_{\lambda} F^{(2)}[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] \\
 &= \cos(\tau + \lambda) F_1[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] d\lambda.
 \end{aligned}$$

Hence, it follows that

$$\begin{aligned}
 \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) &= \int_{\lambda=0}^{\lambda=\omega} [-\cos \bar{\tau} \sin(\tau + \lambda) + \sin \bar{\tau} \cos(\tau + \lambda)] F_1 d\lambda \\
 &= \int_{\lambda=0}^{\lambda=\omega} \sin(\bar{\tau} - \tau - \lambda) F_1 d\lambda \\
 &= \int_{\lambda=0}^{\lambda=\omega} \sin(\omega - \lambda) F_1[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] d\lambda.
 \end{aligned}$$

If we write

$$\omega - \lambda = \lambda',$$

the integral just written is

$$\begin{aligned} & \int_{\lambda'=0}^{\lambda'=\omega} \sin \lambda' F_1 [x, y, \cos (\bar{\tau} - \lambda'), \sin (\bar{\tau} - \lambda')] d\lambda' \\ &= F_1 [x, y, \cos (\bar{\tau} - \lambda_2'), \sin (\bar{\tau} - \lambda_2')] \int_{\lambda'=\omega}^{\lambda'=0} d \cos \lambda', \end{aligned}$$

where  $\lambda_2'$  is intermediary between 0 and  $\omega$ .

We therefore have finally

$$4) \quad \mathfrak{E}(x, y, \bar{p}, q, \bar{p}, \bar{q}) = (1 - \cos \omega) F_1 [x, y, \cos (\bar{\tau} - \lambda_2'), \sin (\bar{\tau} - \lambda_2')].$$

If then  $F_1 [x, y, \cos (\bar{\tau} - \lambda'), \sin (\bar{\tau} - \lambda')]$  has a constant sign between 0 and  $\omega$ , it follows also that  $\mathfrak{E}(x, y, \bar{p}, q, \bar{p}, \bar{q})$  has this sign, since  $\lambda_2'$  is one of the values of  $\lambda'$  within this interval.

The above formula is true for all values of  $\omega$  situated between  $-\pi$  and  $+\pi$ , and since  $\cos (\bar{\tau} - \lambda_2')$  and  $\sin (\bar{\tau} - \lambda_2')$  cannot both be zero at the same time, it is seen that

$$F_1 [x, y, \cos (\bar{\tau} - \lambda_2'), \sin (\bar{\tau} - \lambda_2')] \neq \infty,$$

and consequently the expression 4) for  $\mathfrak{E}$  has not the same defect as the one given in the preceding article.

159. For any displacement of the curve  $\omega \neq 0$ , and consequently  $1 - \cos \omega$  is a positive quantity. Hence  $\mathfrak{E}$  has the same sign as  $F_1$ . If  $F_1 [x, y, \cos (\bar{\tau} - \lambda), \sin (\bar{\tau} - \lambda)]$  is found by examination to have always the same sign independently of  $\cos (\bar{\tau} - \lambda)$ ,  $\sin (\bar{\tau} - \lambda)$  for every point of the curve within the interval in question, then we may be convinced that there is a maximum or a minimum of the integral without the derivation and examination of the function  $\mathfrak{E}$ . By this process, however, we have shown without the second variation that the

function  $F_1(x, y, p, q)$  can change its sign for no point on the curve, and for no direction of the tangent to the curve at a point.

160. It is evident that if  $F_1$ , considered as a function of its third and fourth arguments, has a definite sign, then  $\mathfrak{E}$  has also the same sign; but if  $\mathfrak{E}$  retains a definite sign,  $p$  and  $q$  being fixed while  $\bar{p}$  and  $\bar{q}$  are varied, it does not then follow that  $F_1$  always has a definite sign. This is illustrated in the following example, due to Schwarz:

Let

$$\begin{aligned} F(x, y, x', y') &= a \sqrt{x'^2 + y'^2} + \beta \frac{x' y'^2}{x'^2 + y'^2} \\ &= (a + \beta \cos \tau \sin^2 \tau) \sqrt{x'^2 + y'^2}. \end{aligned}$$

It follows that

$$\begin{aligned} F_1(x, y, x', y') &= \frac{a}{(x'^2 + y'^2)^{3/2}} + 2\beta \frac{x'(x'^2 - 3y'^2)}{(x'^2 + y'^2)^3} \\ &= (a + 2\beta \cos 3\tau) \frac{1}{(x'^2 + y'^2)^{3/2}}, \end{aligned}$$

and, since  $x'^2 + y'^2 = \cos^2 \lambda + \sin^2 \lambda = 1$ ,

$$\begin{aligned} \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) &= \int_{\lambda=0}^{\lambda=\omega} \sin(\omega - \lambda) F_1[x, y, \cos(\tau + \lambda), \sin(\tau + \lambda)] d\lambda \\ &= \int_{\lambda=0}^{\lambda=\omega} \sin(\omega - \lambda) (a + 2\beta \cos 3\lambda) d\lambda, \end{aligned}$$

where we have written  $\tau + \lambda = \lambda$  or  $\tau = 0$ ; i. e., we have taken the  $X$ -axis as the initial direction, from which  $\omega$  is measured.

Noting that

$$\sin(\omega + 2\lambda) + \sin(\omega - 4\lambda) = 2 \sin(\omega - \lambda) \cos 3\lambda,$$

it is seen that

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = (1 - \cos \omega) [a + \beta (\cos \omega + \cos^2 \omega)].$$

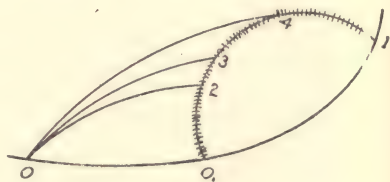
The greatest and least values that  $\cos \omega + \cos^2 \omega$  can have are 2 and  $-\frac{1}{4}$ , the corresponding values of  $\omega$  being 0 and  $\frac{2}{3}\pi$ . Hence, if we make  $a=1$  and  $\beta=1$ , the function  $\mathfrak{E}$  is situated between the values

$$\frac{3}{4}(1 - \cos \omega) \text{ and } 3(1 - \cos \omega),$$

and can consequently vanish only for  $\omega=0$ , and is never negative. On the other hand,  $1 + 2 \cos 3\tau$  changes sign repeatedly, for example, when  $\tau=40^\circ$ .

161. The proof stated at the end of Art. 155 is of paramount importance in the determination whether there exists a true maximum or minimum. The proof of the sufficiency of this theorem, as illustrated in Art. 156, was given in a somewhat different form by Prof. Schwarz. Owing to its importance we add another proof, taken from the lectures of Weierstrass.

Let 00,1 be the curve which satisfies the differential equation  $G=0$ , and let 0,131 be the arbitrary curve in the field, as defined in Art. 156. Let 3 be any point on the arbitrary curve, whose coordinates we consider as functions of length of arc  $s$  (instead of  $t$ , as before). The point 0<sub>1</sub> is taken between 0 and 1 so that the curve 0,131 may lie wholly within the field, since the field might terminate in a point at 0. From the point 0 we draw a curve to 3 which satisfies the differential equation  $G=0$ . We consider the sum of integrals  $I_{03} + \bar{I}_{31}$  as



a function of  $s$ . This function we denote by  $\int(s)$ . Further, take on the arbitrary curve a point 2 in the neighborhood of the point 3 and before it. Join the points 0 and 2 by a curve which satisfies the differential equation  $G=0$ . Then, if we denote the increment of  $s$  by  $\sigma$ , it is seen that

$$\begin{aligned} 5) \quad \int(s - \sigma) - \int(s) &= I_{02} + \bar{I}_{21} - I_{03} - \bar{I}_{31} = I_{02} - I_{03} + \bar{I}_{23} \\ &= \mathfrak{E}(x_3, y_3, p_3, q_3, \bar{p}_3, \bar{q}_3) \sigma + (\sigma)^2. \end{aligned}$$

In the same manner take a point 4 immediately after the point 3 on the arbitrary curve and join this point with the point 0 by a curve which satisfies the differential equation  $G = 0$ . Then we have

$$\begin{aligned} 6) \quad \int(s - \sigma) - \int(s) &= I_{04} - I_{03} - \bar{I}_{34} \\ &= -\bar{\mathfrak{E}}(x_3, y_3, p_3, q_3, \bar{p}_3, \bar{q}_3) \sigma + (\sigma)^2. \end{aligned}$$

It therefore follows that

$$\begin{aligned} 7) \quad \lim_{\sigma \rightarrow 0} \frac{\int(s - \sigma) - \int(s)}{-\sigma} &= \lim_{\sigma \rightarrow 0} \frac{\int(s + \sigma) - \int(s)}{\sigma} \\ &= -\bar{\mathfrak{E}}(x_3, y_3, p_3, q_3, \bar{p}_3, \bar{q}_3); \end{aligned}$$

that is, the quantity  $-\bar{\mathfrak{E}}(x_3, y_3, p_3, q_3, \bar{p}_3, \bar{q}_3)$  is the differential quotient of the function  $\int(s)$  at the point 3.

*If, then, along the curve  $0_131$  the function  $\bar{\mathfrak{E}}$  is nowhere positive, the function  $\int(s)$  continuously diminishes when the point 3 slides from  $0_1$  toward the point 1.*

Let the point  $0_1$ , which was taken very near the point 0, coincide with this point; then we can say:

*If the function  $\bar{\mathfrak{E}}$  is nowhere positive and is not zero at every point of the arbitrary curve  $031$ , the integral taken over the original curve is always greater than the integral extended over the curve  $031$ ; and if the function  $\bar{\mathfrak{E}}$  is not negative and not zero at every point of the curve  $031$ , then the integral taken over the original curve  $01$  is continuously less than the integral extended over the arbitrary curve  $031$ .*

162. It remains yet to see if it is possible for the function  $\bar{\mathfrak{E}}$  to vanish along the whole curve  $031$ . It appears from the formula 3) that this is possible only when along the whole curve we have

$$(\bar{p}q - q\bar{p})^2 = 0, \quad \text{or} \quad \bar{p}q - q\bar{p} = 0.$$

In this case every curve  $03$  which satisfies the differential equation  $G = 0$  has a common tangent at the point 3 with the curve  $031$ .

We shall show that *the curve MN which is formed of the points conjugate to the point 0 has this property, and that no curve having this property can be drawn from 0 within the region that is bounded by MN.* In other words,  $\mathfrak{E}$  is equal to zero along the curve MN, but is not equal to zero for all the points of any other curve that can be drawn within the region that is enveloped by MN.

All the curves that satisfy the differential equation  $G=0$ , which pass through one point, and whose initial directions differ from one another by very small quantities, may be represented (Art. 148) in the form

$$x = \phi(t, k), \quad y = \psi(t, k),$$

where the values of  $k$  are within certain limits.

To each curve corresponds a different value of  $k$ . If, therefore, we fix a value of  $k$  and take a second value  $k+k'$ , the curve which corresponds to this value may be expressed by the equations

$$x + \xi = \phi(t + \tau', k + k'),$$

$$y + \eta = \psi(t + \tau', k + k'),$$

where the same value of  $t$  corresponds to the initial directions of both curves.

If the latter curve is cut by the former, we must have

$$0 = \phi'(t) \tau' + \frac{\partial \phi}{\partial k} k' + (\tau', k')_2,$$

$$0 = \psi'(t) \tau' + \frac{\partial \psi}{\partial k} k' + (\tau', k')_2.$$

The determinant of the linear terms of the equations just written gives, when put equal to zero, the equation for the determination of the point conjugate to the initial point, *i. e.*,

$$\phi'(t) \frac{\partial \psi}{\partial k} - \psi'(t) \frac{\partial \phi}{\partial k} = 0. \quad (14)$$

The smallest root of this equation, which is greater than the value  $t_0$  of  $t$ , gives the value of  $t$ , which belongs to the conjugate point. If this value is  $t_1$ , then the coordinates of the point are

$$\bar{x} = \phi(t_1, k), \quad \bar{y} = \psi(t_1, k).$$

If we consider  $t_1$  as a function of  $k$ , defined through the equation ( $\lambda'$ ), and if we give to  $k$  a series of values, the two equations just written represent the curve that is constituted of the points conjugate to 0.

The direction-cosines of the tangent to this curve are proportional to the quantities  $\frac{d\bar{x}}{dk}, \frac{d\bar{y}}{dk}$ . But we also have

$$\frac{d\bar{x}}{dk} = \frac{\partial \phi(t_1, k)}{\partial t_1} \frac{dt_1}{dk} + \frac{\partial \phi(t_1, k)}{\partial k},$$

$$\frac{d\bar{y}}{dk} = \frac{\partial \psi(t_1, k)}{\partial t_1} \frac{dt_1}{dk} + \frac{\partial \psi(t_1, k)}{\partial k}.$$

Multiply the first of these equations by  $\frac{\partial \psi(t_1, k)}{\partial t_1} = \psi'(t_1)$ , and subtract from it the second after it has been multiplied by  $\frac{\partial \phi(t_1, k)}{\partial t_1} = \phi'(t_1)$ . We have then, with the aid of ( $A$ ),

$$\phi'(t_1) \frac{d\bar{y}}{dk} - \psi'(t_1) \frac{d\bar{x}}{dk} = 0.$$

Since  $\phi'(t_1), \psi'(t_1)$  are proportional to the direction-cosines of the tangent at a point  $t_1$  of the curve through  $t_0$  and  $t_1$ , which satisfies the differential equation  $G=0$ , it follows from the above equation that the tangents to both curves at the point  $t_1$  coincide. *Hence, the locus of the conjugate points to 0 is the envelope of the curves through 0, which satisfy the differential equation  $G=0$ .*

163. Let  $\bar{x}=f(u)$  and  $\bar{y}=g(u)$  be an arbitrary curve 031, which passes through the point 0, and is situated entirely within the region bounded by the envelope. Further, suppose that 031 does not coincide throughout its whole extent with any of the curves passing through 0, which satisfy the differential equation

$G=0$ . Suppose, however, that 031 is touched by the curves that pass through 0 and satisfy the differential equation  $G=0$ . At the point of contact we must have

$$\phi(t, k) = f(u), \quad \psi(t, k) = g(u),$$

and

$$\frac{\partial \phi}{\partial t} \frac{dg}{du} - \frac{\partial \psi}{\partial t} \frac{df}{du} = 0. \quad (B)$$

The values of  $t$  and  $u$ , which belong to the point of contact, are determined as functions of  $k$  through the first two equations.

These equations, being true for sufficiently small values of  $k$ , may be differentiated with respect to  $k$ , and we thus have:

$$\frac{\partial \phi}{\partial t} \frac{dt}{dk} + \frac{\partial \phi}{\partial k} = \frac{df}{du} \frac{du}{dk},$$

$$\frac{\partial \psi}{\partial t} \frac{dt}{dk} + \frac{\partial \psi}{\partial k} = \frac{dg}{du} \frac{du}{dk}.$$

If we multiply the first of these equations by  $\frac{dg}{du}$  and the second by  $-\frac{df}{du}$  and add, we have with the aid of (B)

$$\frac{\partial \phi}{\partial k} \frac{dg}{du} - \frac{\partial \psi}{\partial k} \frac{df}{du} = 0.$$

If between this equation and the equation (B) we eliminate the quantities  $\frac{dg}{du}$  and  $\frac{df}{du}$ , we have

$$\frac{\partial \phi}{\partial t} \frac{\partial \psi}{\partial k} - \frac{\partial \psi}{\partial t} \frac{\partial \phi}{\partial k} = 0,$$

an equation, which served for the determination of the point conjugate to the initial point. Consequently *the point of contact of the curve, that passes through 0 and satisfies the differential equation  $G=0$ , with the arbitrary curve must be the point conjugate to 0.*

But this is possible only if the curve  $\bar{x}=f(u)$ ,  $\bar{y}=g(u)$  coincides with the envelope; while according to our supposition the curve 031 is to lie entirely within the region that is bounded by

the envelope. It follows that there can be within the region no curve 031 such that each of the curves which satisfies the differential equation  $G=0$ , and which joins the point 0 with a point of 031, touches 031 at the same time.

Hence, the quantity  $q\bar{p} - p\bar{q}$  can be everywhere zero only when the arbitrary curve between 0 and 1 coincides throughout its whole extent with one of the curves that passes through 0 and satisfies the differential equation  $G=0$ . But since, within the strip of surface inclosing the field as we have defined it, there can be only one curve drawn through 0 and 1 which satisfies the differential equation  $G=0$ , it follows that the arbitrary curve 031 can coincide only with the original curve 01, and then it is not a variation of that curve. It therefore follows that *the function  $\mathfrak{E}$  cannot vanish for all the points of the curve that has been subjected to variation.*

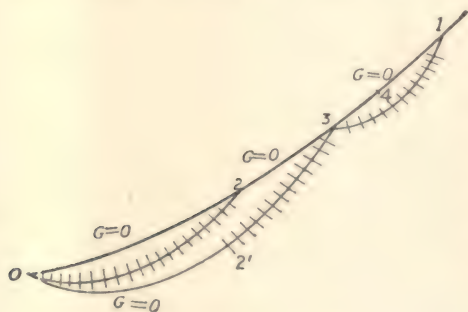
164. It is not necessary that the curve 031 be a single trace of a regular curve in its whole extent. If we assume that 031 is composed of an arbitrary number of regular portions of curve, the integral may be regarded as the sum of the integrals over the single portions, and the conclusions made above are also applicable.

It may happen that one of the portions of curve coincides throughout its whole extent with a portion of one of the curves that goes through 0 and satisfies the differential equation  $G=0$ .

If this is the case for 23, for example, so that  $\mathfrak{E}$  is equal to zero along 23, then we may replace this portion of curve by an arbitrary portion of curve 2'3, which lies very near 23. Then the theorem proved above is true for the curve 02'31, viz., that

$$I_{03} \geq I_{02'} + \bar{I}_{23},$$

according as the function  $\mathfrak{E}$  is nowhere positive or nowhere negative along the curve 02'31. Now, if we bring the curve 2'3 as near



to the curve 23 as we wish, the absolute value of the difference  $I_{03} - I_{02'} - \bar{I}_{2'3}$  can be made smaller than any arbitrarily small quantity  $\delta$ ; and, in accordance with what was proved above, in the first case the difference  $I_{03} - I_{02'} - \bar{I}_{2'3}$  is certainly not *negative*, and in the second case it is not *positive*.

If we shove the point 3 further along the arbitrary curve toward 1, then, when 3 takes a position in the neighborhood of 4, it follows again that  $I_{04} - I_{03} - \bar{I}_{34}$  is greater or less than zero, and, as above, we see that the integral  $I_{01}$ , extended over the curve that satisfies the differential equation  $G = 0$ , is greater or less than the integral taken over the arbitrary curve 0231, according as the function  $\mathfrak{E}$  is nowhere negative or nowhere positive.

165. Further, it is not necessary that the single portions of the curve which has been subjected to variation be regular in order that our conclusions be correctly drawn, if only the coordinates can be expressed as functions of some quantity, and if these functions have derivatives. Finally, if we consider the variation made quite arbitrary, so that only the positions of the points are given, while it is not known whether their coordinates have derivatives, then indeed the integral taken over this curve has no longer any meaning. But the meaning of the integral may be extended so that it has a signification even in this case. For if at first we assume that the coordinates of the curve, which has been subjected to variation, are expressible through functions that have derivatives, then the integral taken over the curve is

$$\int_{t_0}^{t_1} F[f(t), g(t), f'(t), g'(t)] dt.$$

This integral distributed into a sum of integrals (corresponding to the intervals  $t_0 \dots \tau_1, \tau_1 \dots \tau_2, \dots, \tau_n \dots t_1$ ) is equal to

$$\int_{t_0}^{\tau_1} F dt + \int_{\tau_1}^{\tau_2} F dt + \dots + \int_{\tau_n}^{t_1} F dt. \quad (C)$$

We assume that the points  $x_0, y_0; x_1, y_1; \dots x_n, y_n; x_{n+1}, y_{n+1}$  correspond to the values  $t_0, \tau_1, \dots \tau_n, t_1$ .



where we must understand by  $\tau_0$  the value  $t_0$ , and by  $t_{n+1}$  the value  $t_1$ .

Since  $\tau_v - \tau_{v-1}$  are positive quantities, and the functions  $F$  in regard to  $x'_1, y'_1, \dots$  are homogeneous of the first degree, we may write the above limit in the form

$$\lim_{n \rightarrow \infty} \left\{ \sum_{v=1, 2, \dots, n+1} F[x_{v-1}, y_{v-1}, (\tau_v - \tau_{v-1}) x'_{v-1}, (\tau_v - \tau_{v-1}) y'_{v-1}] \right\};$$

or, since

$$x_v - x_{v-1} = (\tau_v - \tau_{v-1}) x'_{v-1} + (\tau_v - \tau_{v-1}) [\tau_v - \tau_{v-1}],$$

the above expression is

$$\lim_{n \rightarrow \infty} \{ F(x_0, y_0, x_1 - x_0, y_1 - y_0) + \dots + F(x_n, y_n, x_{n+1} - x_n, y_{n+1} - y_n) \}.$$

166. The integral in the above form has a more general meaning than the one hitherto employed, with which, however, it coincides in every particular where that one has a meaning. We may assume, with respect to any arbitrary variation, a series of points  $x_0, y_0; x_1, y_1; \dots x_n, y_n; x_{n+1}, y_{n+1}$  of such a nature that the distance between, say, two successive points does not exceed a certain quantity  $\delta$ .

We then form the sum

$$F(x_0, y_0, x_1 - x_0, y_1 - y_0) + \dots + F(x_n, y_n, x_{n+1} - x_n, y_{n+1} - y_n).$$

If we make  $\delta$  smaller and smaller by increasing the number of points, it may happen that this sum approaches a definite limit. We call this limit the value of the integral taken over the curve. It may also happen that the limit does not approach a definite value; for example, it may vacillate between two values. We then say the integral taken over this curve has no meaning.

If we think of the series of points that are taken upon the curve, joined together successively by a broken line, the integral taken over this broken line will approach the same limit as will the integral taken over the curve, if the integral has a meaning.

If, therefore, a curve 01 is given, which satisfies all the conditions that have hitherto been made for a maximum or a minimum, and if this curve varies in an arbitrary manner, then if the integral taken over the curve, which has been subjected to variation, has a meaning as defined above, we may draw a broken line, the integral over which deviates as little as we wish from the integral taken over the curve that has been caused to vary and to which the theorem of Art. 161 is applicable. Consequently we may say, *in the case of a maximum, the integral taken over the curve subjected to variation cannot be greater than the integral taken over the original curve, and in the case of a minimum, it cannot be less than the integral taken over the original curve.*

Since we may make the region as narrow as we wish within which all the variations are to lie, we may assume that upon the curve which has been varied a point 3 lies so near to 01 (but not upon it) that two curves 03, 31 can be drawn between the points 0 and 3 and between 3 and 1, which also satisfy all the conditions of the problem.

For the sake of brevity, let us assume that we have to do with a maximum. Then, as we have just seen, the integrals over 03 and 31 cannot at all events be smaller than the integrals over the corresponding parts of the curve which has been varied; but, after the preceding theorems, the integral taken over 01 is greater than the sum of the integrals taken over 03 and 31, and consequently also greater than the integral over the curve that has been varied. A maximum is therefore in reality present.

167. We may now investigate the behavior of the function  $\mathfrak{E}$  in the case of the four problems which we last considered in Arts. 140-144.

*The problem of the surface of rotation of minimum area.*

We saw that the catenary between limits, within which were situated no pair of conjugate points, was the curve that described a surface of minimum area when rotated around the axis of the half-plane. From the point  $P_0$  we may draw in any direction a curve which satisfies the differential equation  $G = 0$  (a catenary);

the function  $F_1$  is positive for each of these curves as soon as we limit ourselves to the half-plane in which  $y$  is positive. A true minimum will therefore in reality enter. For if  $p, q$  are the direction-cosines of the tangent to the catenary at any point,  $\bar{p}, \bar{q}$  those of the tangent to any arbitrary curve through the same point, then, owing to the relations

$$F^{(1)}(x, y, x', y') = \frac{yx'}{\sqrt{x'^2 + y'^2}}, \quad F^{(2)}(x, y, x', y') = \frac{yy'}{\sqrt{x'^2 + y'^2}},$$

it follows that

$$F^{(1)}(x, y, p, q) = yp, \quad F^{(2)}(x, y, p, q) = yq,$$

since

$$p^2 + q^2 = 1;$$

and consequently

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = y \{ (\bar{p} - p) \bar{p} + (\bar{q} - q) \bar{q} \} = y \{ 1 - (p\bar{p} + q\bar{q}) \}.$$

The expression  $p\bar{p} + q\bar{q}$  is the cosine of the angle between the two tangents. Hence we see that the function  $\mathfrak{E}$  is negative for no point which comes under consideration, and for no two directions  $p, q$  and  $\bar{p}, \bar{q}$ .

If, therefore,  $y=0$  for no point of the curve, our former conclusions are applicable, and a true minimum of the integral has, in reality, been found.

168. *The Brachistochrone.* We saw that this curve is the cycloid

$$x = g + r(1 - \sin t),$$

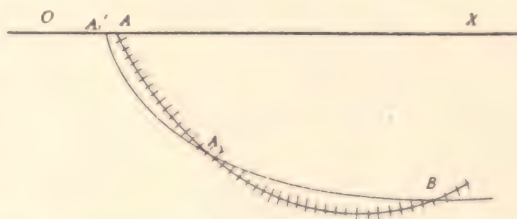
$$y + a = r(1 - \cos t).$$

We assume that the point  $A$ , from which the moving point starts, having an initial velocity proportional to the quantity  $\sqrt{a}$ , is the origin of coordinates, and that the  $Y$ -axis is the direction of gravity. We saw that the cycloid could then be generated by a point described by a circle which rolls upon the straight line  $y = -a$ . If  $a$  is different from zero, an arc of a cycloid may be constructed through  $A$  in any direction. If the curve passes

through a singular point it does not minimize the integral, as was shown in Art. 104. If  $A$  and  $B$  are not singular points, the function  $F_1$  has a positive value different from zero everywhere along this curve and in the neighborhood of it in every direction.

Between two arbitrary points (see Art. 105), when the quantity  $a$  is given, there can always be drawn one, and only one, arc of a cycloid which has no singular points between these two points. If, therefore,  $a$  is different from zero, and consequently  $A$  and  $B$  are not singular points, then (see Art. 159) it follows that the curve, in reality, causes the integral to have a minimum value. Suppose that  $A$  or  $B$  is a singular point; then at this point  $F_1$  becomes infinite, a case which we consider in the next Article.

169. Suppose  $A$  is a singular point and  $a=0$ . Draw an arbitrary curve between  $A$  and  $B$ . Take upon this curve in the neighborhood of  $A$  a point  $A_1$ , and through  $A_1$  and  $B$  draw a cycloid which cuts the  $X$ -axis at  $A_1'$ . The material point under the action of gravity passes through  $A_1$  with the same velocity which it would have at an equal distance below the  $X$ -axis if it traversed the cycloid drawn through  $A$  and  $B$ .



The following notation may be introduced:

$I_{01}$  to denote the time of falling between  $A_1'$  and  $B$  upon the cycloid  $A_1'B$ ,

$I_{01}'$  to denote the time of falling between  $A$  and  $A_1$  upon the arbitrary curve  $AB$ ,

$I$  to denote the time of falling between  $A_1$  and  $B$  upon the cycloid  $A_1B$ ,

$I'$  to denote the time of falling between  $A_1$  and  $B$  upon the arbitrary curve  $A_1B$ .

We proved that

$$I' > I,$$

and therefore, if we write

$$\bar{I} = I_{01}' + I',$$

it follows that

$$\bar{I} > I + I_{01}'.$$

Now, let the point  $A_1$  approach nearer and nearer the point  $A$ , so that the integral  $I$  approaches the limit  $I_{01}$ , while  $I_{01}'$  becomes indefinitely small. We must then have

$$\bar{I} \geq I_{01}.$$

That  $\bar{I}$  is *greater* than  $I_{01}$  may be seen as follows: As soon as  $G \neq 0$  along a portion of curve, we may always vary it in such a way that the increment in the corresponding integral may have any sign. If, then,  $G \neq 0$  along the whole curve  $AA_1B$ , we may substitute another curve, for which, if  $I''$  is the value of the integral which belongs to it,

$$I'' < \bar{I}.$$

But since we also have

$$I'' \geq I_{01},$$

it follows that

$$\bar{I} > I_{01}.$$

If, on the other hand,  $G=0$  along the whole curve  $AA_1B$ , then this curve must consist of several cycloidal arcs; since, if it were only one, the curves  $AA_1B$  and  $AB$  would be identical. These arcs must have different tangents at the point where they come together; for, since this point cannot lie on the  $X$ -axis, a consecutive point having the same direction must lie on the same cycloidal arc. If corners were present, however, they could be so rounded off that there would be a shorter path between the two points, and consequently, the velocity being the same, the time of falling would be shorter.

Hence the arc of a cycloid also minimizes the time of falling between  $A$  and  $B$  in the case where  $A$  is a singular point; that is, when the material point starts from  $A$  with an initial velocity that is zero.

The conclusions just made are also applicable, if  $B$  is a singular point; for it makes no difference whether the material point ascends from  $B$  to  $A$  or falls from  $A$  to  $B$ , if we allow the material point to go back with the same initial velocity with which it arrived at  $B$ . On the way back it will reach  $A$  with its original velocity. Its velocity will be the same in both cases at all points of the curve, but directed toward opposite directions. The integral taken over the curve has the same value in both cases; and consequently the curve which caused the integral to have a minimum value will also, in the second case, minimize the integral.

170. *The problem of the geodesic line on a sphere* offers here nothing of special interest. It is found that the function  $\mathfrak{E}$  retains a positive sign along the arc of a great circle situated between two poles.

171. *Problem of the surface of revolution which offers the least resistance.*

In this problem

$$F(x, y, x', y') = \frac{x x'^3}{x'^2 + y'^2},$$

and since

$$p^2 + q^2 = 1,$$

it follows that

$$F(x, y, p, q) = \frac{x p^3}{p^2 + q^2} = x p^3,$$

$$\frac{\partial F}{\partial p} = x(p^4 + 3p^2 q^2),$$

$$\frac{\partial F}{\partial q} = -2 x p^3 q.$$

Substituting these values in

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = F(x, y, \bar{p}, \bar{q}) - \bar{p} \frac{\partial F}{\partial p} - \bar{q} \frac{\partial F}{\partial q},$$

we have

$$\begin{aligned} \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) &= x[\bar{p}^3 - \bar{p}p^2 - 2\bar{p}p^2q^2 + 2\bar{q}p^3q] \\ &= x[\bar{p}(\bar{p}^2 - p^2) + 2p^2q(\bar{p}q - \bar{p}q)] \\ &= x[\bar{p}\{p^2(p^2 + q^2) - p^2(\bar{p}^2 + \bar{q}^2)\} + 2p^2q(\bar{p}q - \bar{p}q)] \\ &= x(\bar{p}q - p\bar{q})[\bar{p}(\bar{p}q + p\bar{q}) - 2p^2q] \\ &= x(\bar{p}q - p\bar{q})[\bar{p}(\bar{p}q - p\bar{q})(p^2 + q^2) + 2p\bar{p}\bar{q}(p^2 + q^2) \\ &\quad - 2p^2q(\bar{p}^2 + \bar{q}^2)] \\ &= x(\bar{p}q - p\bar{q})[\bar{p}(\bar{p}q - p\bar{q})(p^2 + q^2) - 2p^2\bar{p}(\bar{p}q - p\bar{q}) \\ &\quad + 2pq\bar{q}(\bar{p}q - p\bar{q})] \\ &= x(\bar{p}q - p\bar{q})^2[\bar{p}(p^2 + q^2) - 2p^2\bar{p} + 2pq\bar{q}] \\ &= x(\bar{p}q - p\bar{q})^2[\bar{p}(q^2 - p^2) + 2pq\bar{q}]. \end{aligned}$$

Writing

$$\cos \bar{\tau} = \bar{p}, \quad \sin \bar{\tau} = \bar{q}, \quad \cos \tau = p, \quad \sin \tau = q,$$

we have

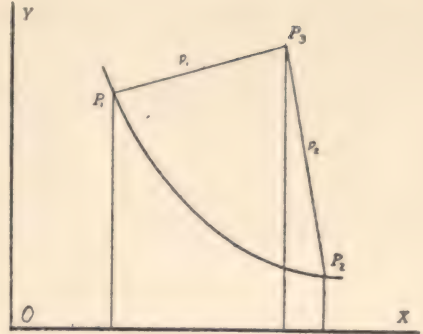
$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = -x \sin(\bar{\tau} - \tau)^2 \cos(\bar{\tau} + \tau).$$

Therefore the sign of  $\mathfrak{E}$  is the same as that of  $-\cos(\bar{\tau} + \tau)$ , and may be either positive or negative by properly choosing  $\bar{\tau}$ , an angle which depends upon  $\bar{p}, \bar{q}$ .

*At every point of the curve for which  $x \neq 0$  the function  $\mathfrak{E}$  can have different signs, and consequently a maximum or a minimum value of the integral does not exist. We saw in Art. 109 that  $x$  must be different from zero for all points of the arc.*

172. Legendre (*Mémoire sur la manière de distinguer les maxima des minima dans le Calcul des Variations*) showed that by taking a zigzag line for the generating curve, the resistance could be made as small as we wish.

Suppose that the arc  $P_1P_2$  had the desired property of generating a surface of least resistance, and suppose that the tangent to this curve is nowhere parallel to the  $X$ -axis. Writing  $p = \frac{dx}{dy}$ , it follows that  $p \neq 0$  along the arc  $P_1P_2$ .



We have then (Art. 108)

$$I_{1,2} = \int_{t_1}^{t_2} \frac{xx'^3}{x'^2 + y'^2} dt = \int_{x_1}^{x_2} \frac{p^2 x}{1 + p^2} dx.$$

Since  $p$  is finite and continuous along the arc in question, it follows that  $\frac{p^2}{1 + p^2}$  has the same properties along the arc, and therefore

$$I_{1,2} = \frac{p_0^2}{1 + p_0^2} \int_{x_1}^{x_2} x dx = \frac{p_0^2}{1 + p_0^2} \frac{x_2^2 - x_1^2}{2},$$

where  $p_0$  is a mean value of  $p$ , lying between the points  $P_1$  and  $P_2$  of the curve.

Between the ordinates at  $P_1$  and  $P_2$  draw a line parallel to the  $Y$ -axis, and on this line take a point  $P_3$  whose ordinate is longer than those of the points  $P_1$  and  $P_2$ . Draw the straight lines  $P_1P_3$  and  $P_2P_3$ , and let  $p_1$  and  $p_2$  be the values of  $\frac{dx}{dy}$  for these lines. The integral  $\int F dt$  taken over the broken line  $P_1P_3P_2$  may be denoted by  $I_{13} + I_{32}$ , where

$$I_{13} = \int_{x_1}^{x_3} \frac{p_1^2 x dx}{1 + p_1^2} = \frac{p_1^2}{1 + p_1^2} \frac{(x_3^2 - x_1^2)}{2}$$

and

$$I_{32} = \int_{x_3}^{x_2} \frac{p_2^2 x dx}{1 + p_2^2} = \frac{p_2^2}{1 + p_2^2} \frac{x_2^2 - x_3^2}{2}.$$

We have then

$$\begin{aligned} I_{132} - I_{12} &= I_{13} + I_{32} - I_{12} \\ &= \frac{p_1^2(x_3^2 - x_1^2)}{2(1 + p_1^2)} + \frac{p_2^2(x_2^2 - x_3^2)}{2(1 + p_2^2)} - \frac{p_0^2(x_2^2 - x_1^2)}{2(1 + p_0^2)}. \end{aligned}$$

The first two terms of this expression may be made as small as we choose by sufficiently diminishing the quantities  $p_1$  and  $p_2$ , which is done by removing indefinitely the point  $P_3$  along its ordinate. Hence, their sum is less than the third term, so that, consequently,

$$I_{132} < I_{12}.$$

This result may also be derived as follows:

$$\begin{aligned} \frac{I_{13} + I_{32}}{I_{12}} &= \frac{p_1^2(1 + p_0^2)}{p_0^2(1 + p_1^2)} \frac{x_3^2 - x_1^2}{x_2^2 - x_1^2} + \frac{p_2^2(1 + p_0^2)}{p_0^2(1 + p_2^2)} \frac{x_2^2 - x_3^2}{x_2^2 - x_1^2} \\ &< \frac{p_1^2(1 + p_0^2)}{p_0^2(1 + p_1^2)} + \frac{p_2^2(1 + p_0^2)}{p_0^2(1 + p_2^2)}, \end{aligned}$$

since  $x_1 < x_3 < x_2$ .

Hence also for a greater reason

$$\frac{I_{13} + I_{32}}{I_{12}} < (p_1^2 + p_2^2) \frac{1 + p_0^2}{p_0^2}.$$

From this it is seen that the ratio  $\frac{I_{13} + I_{32}}{I_{12}}$  may be indefinitely diminished by properly choosing  $p_1$  and  $p_2$ . There is then no limit to the least possible resistance.

The method just given does not replace the  $\mathfrak{E}$ -criterion which shows that no surface of minimal resistance exists. It shows simply that no rotational surface exists, which gives an absolute

minimum of resistance—a resistance less than any other neighboring surface. The  $\mathfrak{E}$ -criterion shows that no minimum exists in the sense of giving a resistance less than that given by any neighboring curve within a limited neighborhood.

173. In the general case, when  $F(x, y, x', y')$  is a rational function of  $x'$  and  $y'$ , neither a maximum nor a minimum can exist. For in this case

$$\mathfrak{E} = F(x, y, \bar{p}, \bar{q}) - \bar{p} \frac{\partial F}{\partial p} - \bar{q} \frac{\partial F}{\partial q}$$

is also a rational function of  $\bar{p}$  and  $\bar{q}$  and homogeneous in these quantities of the first degree. Consequently,

$$\mathfrak{E}(x, y, k\bar{p}, k\bar{q}) = k \mathfrak{E}(x, y, \bar{p}, \bar{q}),$$

and therefore

$$\mathfrak{E}(x, y, -\bar{p}, -\bar{q}) = -\mathfrak{E}(x, y, \bar{p}, \bar{q}).$$

It is thus seen that we have only to reverse the direction of the displacement to effect a change of sign in the function  $\mathfrak{E}$ .

174. We have now completely solved the four problems that were proposed in Chapter I, and at the same time one of the principal parts of the Calculus of Variations has been finished. After stating succinctly the four criteria that have been established, we shall take up the second part, which has as its object the theoretical and practical solution of problems, a general type of which were the Problems V and VI of Chapter I.

These criteria may be summarized as follows (cf. Art. 125):  
*There exists a minimum or a maximum value of the integral*

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt,$$

where  $F$  is a one-valued, regular function of its four arguments and homogeneous of the first degree in  $x'$  and  $y'$ , if

- 1) the differential equation  $G=0$  is satisfied for every point of the curve;

- 2)  $F_1$  is positive or negative throughout the whole interval  $t_0 \dots t_1$ ;
- 3) there are no conjugate points of the curve within the interval  $t_0 \dots t_1$  (limits included);
- 4) the function  $\mathfrak{E}$  is positive or negative throughout the whole interval  $t_0 \dots t_1$ .

In this discussion we have excluded the cases where

- 1) the extremities of the curve are conjugate points;
- 2)  $F_1=0$  for some point of the curve;
- 3)  $F_1=0$  for some stretch of the curve;
- 4)  $\mathfrak{E}=0$  for some point or stretch of the curve.

A general treatment of the first three cases would require the extension of the theory to variations of a higher order. Otherwise particular devices must be employed in every example in which one of the above exceptional cases is found.

175. Before we begin the consideration of *Relative Maxima and Minima*, we may, at least, indicate the natural extensions and generalizations of the theory which has already been presented: Instead of the determination of a *structure of the first kind\** in the domain of two quantities, it may be required to determine a *structure of the first kind in the domain of  $n$  quantities*.

If a structure of the first kind is determined in the domain of the  $n$  quantities  $x_1, x_2, \dots, x_n$ , then  $n-1$  of these quantities may be expressed as functions of the remaining one, say,  $x_1$ .

Writing

$$u = \int F \left( x_1, x_2, \dots, x_n, \frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}, \dots, \frac{dx_n}{dx_1} \right) dx_1,$$

---

\* See my Lectures on the Theory of Maxima and Minima of Functions of Several Variables, pp. 15 and 86.

it is seen that  $u$  is so connected with the  $n-1$  functions that

$$\frac{du}{dx_1} = F\left(x_1, x_2, \dots, x_n, \frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}, \dots, \frac{dx_n}{dx_1}\right).$$

The difference of the values of  $u$  at the initial-point and at the end-point of the structure is expressed by a definite integral.

This integral takes the form, when we consider the  $x$ 's expressed as functions of  $t$ , say,  $x_1=x_1(t)$ ,  $x_2=x_2(t)$ , ...,  $x_n=x_n(t)$ ,

$$I = \int_{t_0}^{t_1} F(x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n) dt.$$

The function  $F$  must be a one-valued, regular function of its arguments in the whole or a limited portion of the fixed domain.

The value of the integral  $I$  is independent of the manner in which the variables  $x_1, x_2, \dots, x_n$  have been expressed as functions of  $t$ . It therefore follows after the analogon of Art. 68 that the function  $F$  is subjected to the further restriction:

$$\begin{aligned} kF(x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n) \\ = F(x_1, x_2, \dots, x_n, kx'_1, kx'_2, \dots, kx'_n), \end{aligned}$$

where  $k$  is a positive constant.

The indicated generalization of the problem given in Art. 13 may accordingly be expressed as follows:

*The  $n$  quantities  $x_1, x_2, \dots, x_n$  are to be determined as functions of a quantity  $t$  in such a manner that for the analytical structure that is defined through the equations*

$$x_1=x_1(t), x_2=x_2(t), \dots, x_n=x_n(t),$$

*the value of the integral*

$$I = \int_{t_0}^{t_1} F(x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n) dt$$

is a maximum or a minimum; in other words, if one causes the above analytical structure to vary indefinitely little, the change in the integral thereby produced must in the case of a maximum be constantly negative, and in the case of a minimum it must be constantly positive. Further, the function  $F$  is to be considered a one-valued, regular function of its arguments, and indeed, with respect to  $x_1', x_2', \dots, x_n'$ , a homogeneous function of the first degree.

176. The treatment of the above problem is found to be the complete analogon of the problem given in Art. 13. A greater complication arises when there are present equations of condition among the variables  $x_1, x_2, \dots, x_n$ . An example of this kind we had in Problem III of Chapter I.

This problem may be expressed thus: *Among all the curves in space which belong to the surface*

$$f(x, y, z) = 0,$$

*determine that one for which the integral*

$$\int_{t_0}^{t_1} \sqrt{x'^2 + y'^2 + z'^2} dt$$

*is a minimum.*

The general problem may be formulated as follows: *Among the structures of the first kind in the domain of the quantities  $x_1, x_2, \dots, x_n$ , for which the  $m$  equations*

$$f_\mu(x_1, x_2, \dots, x_n) = 0 \quad (\mu = 1, 2, \dots, m; m < n-1)$$

*exist, that one is to be determined for which the integral*

$$I = \int_{t_0}^{t_1} F(x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n') dt$$

*is a maximum or a minimum.*

This problem may be reduced to the one of the preceding Article in an analogous manner as is done in Art. 10 for the case of the shortest line upon a surface. The  $m$  equations of condition may be satisfied by introducing for the variables  $x_1, x_2, \dots, x_n$  functions of  $n-m$  new variables after the method given in the Lectures on the Theory of Maxima and Minima, etc., Chapter I, Art. 15. The new variables are independent of one another, so that the above integral may be replaced by one in which the variables are free from extraneous conditions; or we may proceed as was done in the Theory of Maxima and Minima where the variables are subject to subsidiary conditions (loc. cit., p. 54).

177. The more general problem of the Calculus of Variations, in so far as it has to do with the structures of the first kind, may be stated as follows:

*Among the structures of the first kind in the domain of the  $n$  quantities  $x_1, x_2, \dots, x_n$ , for which definite equations of condition exist, not only among the  $n$  quantities themselves, but also among their first derivatives, that structure is to be determined for which the integral*

$$I = \int_{t_0}^{t_1} F(x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n) dt$$

*becomes a maximum or a minimum*

It may be easily shown that the apparently more general case in which  $F$  is a function of  $x_1, x_2, \dots, x_n$  and of the first and higher derivatives of these quantities, is contained in the problem just stated. For the sake of simplicity, take the case where only two variables are involved and write

$$u = \int_{x_1}^{x_2} F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) dx.$$

If in this integral we express  $x$  and  $y$  as functions of  $t$ , we have

$$\frac{dy}{dt} = \frac{dy}{dx} x', \quad \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} x'^2 + \frac{dy}{dx} x''.$$

We may consequently change the integral  $u$  into

$$u = \int_{t_0}^{t_1} F(x, y, x', y', x'', y'', \dots) dt.$$

We further have

$$\frac{dx'}{dt} - x'' = 0, \quad \frac{dy'}{dt} - y'' = 0.$$

We may therefore write

$$u = \int_{t_0}^{t_1} F\left(x, y, x_1, y_1, \frac{dx_1}{dt}, \frac{dy_1}{dt}\right) dt,$$

with the equations of condition:

$$\frac{dx}{dt} = x_1, \quad \frac{dy}{dt} = y_1.$$

If, then, there appear in  $F$  only the first and second derivatives, it is seen that  $F$  depends upon the four functions  $x, y, x_1, y_1$  which are to be determined, while at the same time the two equations of condition just written must be satisfied. One of the classes of problems belonging to the general problem just stated is the one which was formulated in Art. 17 and which is treated in the following Chapters.

178. It may be mentioned finally that the problem of the Calculus of Variations may be further generalized, if we require the determination of structures of a higher kind. For example,

in the simplest case the three quantities  $x, y, z$  may be determined as functions of two independent variables  $u$  and  $v$ . We have then instead of the single integral the double integral

$$\iint F\left(x, y, z, \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) du dv,$$

which must be a maximum or a minimum.

The treatment of this problem would give a theory of *Minimal Surfaces*.

## Relative Maxima and Minima.

### CHAPTER XIII.

#### STATEMENT OF THE PROBLEM. DERIVATION OF THE NECESSARY CONDITIONS.

179. The nature of many problems which arise in the Calculus of Variations presents subsidiary conditions which limit the arbitrariness that we have hitherto employed in the indefinitely small variations of the analytical structure. Such problems are the most difficult and at the same time the most interesting that occur. These last conditions which enter into the requirement for a maximum or a minimum are in general of a double nature. On the one hand, it may be proposed that among the variables there are to exist equations of condition, as indicated in Arts. 176 and 177. On the other hand, we may require that the maximum or the minimum in question satisfy a further condition, viz., it must cause another given integral to have a prescribed value. Such cases are usually called *Relative Maxima and Minima*.

If we limit our discussion to the region of two variables, then the problem which we have to consider may be expressed as follows (cf. Art. 17):

*Let  $F^{(0)}(x, y, x', y')$  and  $F^{(1)}(x, y, x', y')$  be two functions of the same nature as the function  $F(x, y, x', y')$  hitherto treated. The variables  $x$  and  $y$  are to be so determined as one-valued functions of  $t$  that the curve defined through the equations  $x=x(t)$ ,  $y=y(t)$  will cause the integral*

$$1) \quad I^{(0)} = \int_{t_0}^{t_1} F^{(0)}(x, y, x', y') dt$$

to be a maximum or a minimum, while at the same time for the same equations the integral

$$2) \quad I^{(1)} = \int_{t_0}^{t_1} F^{(1)}(x, y, x', y') dt$$

will have a prescribed value; that is, for every indefinitely small variation of the curve for which the second integral retains its sign unaltered, the first integral, according as a maximum or a minimum is to enter, must be continuously smaller or continuously greater than it is for the curve  $x=x(t)$ ,  $y=y(t)$ .

180. We must first show that it is possible to represent analytically the variations of a curve for which the integral  $I^{(1)}$  retains a constant value.

In the place of the variables  $x, y$  let us make the substitution  $x + \bar{\xi}, y + \bar{\eta}$ . The variation of the second integral is accordingly

$$3) \quad \Delta I^{(1)} = \int_{t_0}^{t_1} G^{(1)} \bar{w} dt + \int_{t_0}^{t_1} (\bar{\xi}, \bar{\eta}, \bar{\xi}', \bar{\eta}')_2 dt,$$

where  $(\bar{\xi}, \bar{\eta}, \bar{\xi}', \bar{\eta}')_2$  denotes that the terms within the brackets are of the second and higher dimensions in  $\bar{\xi}, \bar{\eta}, \bar{\xi}', \bar{\eta}'$ .

We have so to determine  $\bar{\xi}$  and  $\bar{\eta}$  that  $\Delta I^{(1)} = 0$ . For this purpose we write

$$4) \quad \begin{cases} \bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1 + \epsilon_2 \xi_2 + \dots, \\ \bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1 + \epsilon_2 \eta_2 + \dots, \end{cases}$$

where  $\epsilon, \epsilon_1, \dots$  are arbitrary constants and the functions  $\xi, \xi_1, \dots, \eta, \eta_1, \dots$  are functions similar to the quantities  $\xi, \eta$  of the preceding Chapters and vanish for  $t=t_0$  and  $t=t_1$ . Now write

$$w_1 = y' \xi_1 - x' \eta_1$$

and

$$\bar{w} = y' \bar{\xi} - x' \bar{\eta} = \epsilon w + \epsilon_1 w_1 + \epsilon_2 w_2 + \dots$$

Hence, from 3) we have

$$\Delta I^{(1)} = \epsilon \int_{t_0}^{t_1} G^{(1)} w dt + \epsilon_1 \int_{t_0}^{t_1} G^{(1)} w_1 dt + \dots + (\epsilon, \epsilon_1, \dots)_2.$$

If we write

$$5) \quad W_i^{(1)} = \int_{t_0}^{t_1} G^{(1)} w_i dt,$$

it follows that

$$\Delta I^{(1)} = \epsilon W^{(1)} + \epsilon_1 W_1^{(1)} + \epsilon_2 W_2^{(1)} + \dots + (\epsilon, \epsilon_1, \epsilon_2, \dots)_2.$$

The functions  $W_i^{(1)}$  are completely determined as soon as definite values are given to  $\xi, \xi_1, \dots$ ; and, in order that  $\Delta I^{(1)} = 0$ , it is necessary that

$$(A) \quad \epsilon W^{(1)} + \epsilon_1 W_1^{(1)} + \epsilon_2 W_2^{(1)} + \dots + (\epsilon, \epsilon_1, \epsilon_2, \dots)_2 = 0.$$

If any of the quantities  $W_i^{(1)}$ , for example  $W_\lambda^{(1)}$ , are different from zero, we are able to express  $\epsilon_\lambda$  in a power-series of the remaining  $\epsilon$ 's, when these quantities have been chosen sufficiently small.\* The equation  $\Delta I^{(1)} = 0$  may consequently be satisfied for sufficiently small systems of values of the  $\epsilon$ 's.

Substitute one of these systems of values in 4) and it is seen that indefinitely small variations of the curve  $x = x(t), y = y(t)$  exist for which the integral  $I^{(1)}$  remains unaltered. These variations may be analytically represented (see the next Article).

This proof is deficient in the case where all the quantities  $W^{(1)}, W_1^{(1)}, \dots$  are zero for all values of  $\xi_i, \eta_i$ , however  $\xi_i, \eta_i$  may have been chosen. When this is the case,  $G^{(1)}$  must be zero along the whole curve. But this is one of the necessary conditions that the integral  $I^{(1)}$  have a maximum or a minimum value.

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\* Cf. Lectures on the Theory of Maxima and Minima, etc., p. 20.

If, then, for the curve which is derived through the solution of the differential equation  $G^{(0)}=0$  there also enters a maximum or a minimum value of the integral  $I^{(1)}$ , and consequently  $G^{(1)}=0$ , it is in general not possible so to vary the curve that the second integral remains unaltered.

This case is excluded from the present discussion, and is left for special investigation in each particular problem.

181. Let us limit ourselves for the present to the simplest case where

$$\bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1,$$

$$\bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1;$$

and if we denote the integrals in the expansion of  $\Delta I^{(1)}$  that are associated with the coefficients  $\epsilon^i \epsilon_1^j$  by  $W_{ij}^{(1)}$ , the equation corresponding to  $(A)$  of the last article is

$$(A^a) \quad 0 = \epsilon W_{10}^{(1)} + \epsilon_1 W_{01}^{(1)} + \epsilon^2 W_{20}^{(1)} + \epsilon \epsilon_1 W_{11}^{(1)} + \epsilon_1^2 W_{02}^{(1)} + \dots,$$

which series we suppose convergent for sufficiently small values of  $\epsilon$  and  $\epsilon_1$ .

Suppose next we express  $\epsilon_1$  in terms of  $\epsilon$  by the series

$$(B) \quad \epsilon_1 = h_1 \epsilon + h_2 \epsilon^2 + h_3 \epsilon^3 + \dots$$

Then, when this value of  $\epsilon_1$  is substituted in  $(A^a)$ , by equating the coefficients of the different powers of  $\epsilon$  to zero, we have

$$W_{10}^{(1)} + h_1 W_{01}^{(1)} = 0,$$

$$W_{20}^{(1)} + h_1 W_{11}^{(1)} + h_1^2 W_{02}^{(1)} + h_2 W_{01}^{(1)} = 0,$$

$$\dots \dots \dots$$

Hence, denoting the quotients  $-\frac{W_{ij}^{(1)}}{W_{01}^{(1)}}$  by  $V_{ij}$ , where  $W_{01}^{(1)} \neq 0$ , we have

$$h_1 = V_{10},$$

$$h_2 = V_{20} + h_1 V_{11} + h_1^2 V_{02},$$

$$\dots \dots \dots$$

Further, the equation ( $A^a$ ) may be written

$$(A^b) \quad \epsilon_1 = \epsilon V_{10} + \epsilon^2 V_{20} + \epsilon \epsilon_1 V_{11} + \epsilon_1^2 V_{02} + \dots$$

Let us compare this series with the series

$$(C) \quad \epsilon_1 = \frac{g}{1 - \left( \frac{\epsilon}{r} + \frac{\epsilon_1}{r_1} \right)} - g - g \frac{\epsilon_1}{r_1} \\ = g \left[ \frac{\epsilon}{r} + \left( \frac{\epsilon}{r} + \frac{\epsilon_1}{r_1} \right)^2 + \left( \frac{\epsilon}{r} + \frac{\epsilon_1}{r_1} \right)^3 + \dots \right].$$

Suppose from this series we have  $\epsilon_1$  expressed in terms of  $\epsilon$  in the form

$$(B^b) \quad \epsilon_1 = h_1' \epsilon + h_2' \epsilon^2 + h_3' \epsilon^3 + \dots,$$

where the  $h$ 's have been derived from the coefficients of powers of  $\epsilon$  and  $\epsilon_1$  as the  $h$ 's in ( $B$ ) are formed from the coefficients  $V$  in ( $A^b$ ).

The series ( $B^b$ ) is convergent for

$$\left| \frac{\epsilon}{r} \right| + \left| \frac{\epsilon_1}{r_1} \right| < 1.$$

If, then, the coefficients  $V$  of ( $A^b$ ) are in absolute value less than the corresponding coefficients in ( $C$ ), the coefficients  $h$  in ( $B$ ) are less in absolute value than the coefficients  $h'$  in ( $B^b$ ), and therefore the series ( $B$ ) is convergent.

Now the coefficients of  $\epsilon^k \epsilon_1^\mu$  in ( $A^b$ ) and ( $C$ ) are respectively

$$V_{k, \mu} \quad \text{and} \quad \binom{k + \mu}{k} \frac{g}{r^k r_1^\mu},$$

where the symbol  $\binom{m}{n}$  denotes  $\frac{m \cdot m-1 \dots m-n+1}{n!}$ . Hence, for sufficiently small values of  $r$  and  $r_1$ , if

$$\left| \frac{\epsilon}{r} \right| + \left| \frac{\epsilon_1}{r_1} \right| < 1$$

and

$$V_{k, \mu} < \left( \frac{k + \mu}{k} \right) \frac{g}{r^k r_1^\mu},$$

the series ( $B$ ) is convergent, and when substituted in the expression for  $\Delta I^{(1)}$  causes this expression to vanish.

182. The expression for  $\epsilon_1$  as a function of  $\epsilon$  is had from the relation

$$\epsilon_1 = \frac{g}{1 - \left( \frac{\epsilon}{r} + \frac{\epsilon_1}{r_1} \right)} - g - g \frac{\epsilon_1}{r_1}.$$

Hence, it follows that

$$\left( \frac{\epsilon_1}{r_1} \right)^2 + \left( \frac{\epsilon}{r} - \frac{r_1}{r_1 + g} \right) \frac{\epsilon_1}{r_1} + \frac{g\epsilon}{r(r_1 + g)} = 0,$$

or

$$\frac{\epsilon_1}{r_1} = \frac{1}{2} \left[ \frac{r_1}{r_1 + g} - \frac{\epsilon}{r} \pm \sqrt{\left( \frac{\epsilon}{r} - \frac{r_1}{r_1 + g} \right)^2 - \frac{4g\epsilon}{r(r_1 + g)}} \right].$$

Of the two roots we choose the one with the lower sign in order that  $\epsilon_1$  equal zero with  $\epsilon$ . This root may be written

$$\begin{aligned} \frac{\epsilon_1}{r_1} = \frac{1}{2} & \left[ \frac{r_1}{r_1 + g} - \frac{\epsilon}{r} \right. \\ & \left. - \left( \frac{r_1}{r_1 + g} - \frac{\epsilon}{r} \right) \sqrt{1 - \left( \frac{4g\epsilon}{r(r_1 + g)} \right) / \left( \frac{r_1}{r_1 + g} - \frac{\epsilon}{r} \right)^2} \right]. \end{aligned}$$

It is seen that the expression under the radical is finite, continuous and one-valued for values of  $\epsilon$  such that

$$\frac{\epsilon}{r} < \frac{r_1}{r_1 + g} \quad \text{and} \quad \frac{4g\epsilon}{r(r_1 + g)} < \left( \frac{r_1}{r_1 + g} - \frac{\epsilon}{r} \right)^2.$$

183. Returning to the substitutions

$$\bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1,$$

$$\bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1,$$

we assume that the functions  $\xi$ ,  $\xi_1$ ,  $\eta$ ,  $\eta_1$  become zero at the end-points (or limits) of the curve and are so chosen that  $W_{01}^{(1)}$  does

not vanish within the limits of integration. We have then at once from ( $A^a$ ) the power-series

$$\epsilon_1 = -\frac{W_{10}^{(1)}}{W_{01}^{(1)}} \epsilon + \epsilon P(\epsilon),$$

where the power-series  $P(\epsilon)$  vanishes with  $\epsilon$ .

From this we have

$$6) \quad \begin{cases} \bar{\xi} = \epsilon \left( \xi - \frac{W_{10}^{(1)}}{W_{01}^{(1)}} \xi_1 \right) + \epsilon \xi_1 P(\epsilon), \\ \bar{\eta} = \epsilon \left( \eta - \frac{W_{10}^{(1)}}{W_{01}^{(1)}} \eta_1 \right) + \epsilon \eta_1 P(\epsilon). \end{cases}$$

If we subject the integral  $I^{(0)}$  to the same variation, we have [cf. formula ( $A^a$ )]

$$\Delta I^{(0)} = \epsilon W_{10}^{(0)} + \epsilon_1 W_{01}^{(0)} + (\epsilon, \epsilon_1)_2,$$

and consequently

$$\Delta I^{(0)} = \epsilon \left( W_{10}^{(0)} - \frac{W_{10}^{(1)}}{W_{01}^{(1)}} W_{01}^{(0)} \right) + (\epsilon)_2.$$

*If, then, the integral  $I^{(0)}$  is to have a maximum or a minimum value, it is necessary that*

$$W_{10}^{(0)} - \frac{W_{10}^{(1)}}{W_{01}^{(1)}} W_{01}^{(0)}$$

*be equal to zero.*

We have, therefore, the necessary condition

$$\frac{W_{10}^{(0)}}{W_{10}^{(1)}} = \frac{W_{01}^{(0)}}{W_{01}^{(1)}}.$$

From this it is seen that the quotient  $\frac{W_{10}^{(0)}}{W_{10}^{(1)}}$  is independent of the arbitrary functions  $\xi, \eta$ , since it does not vary if we write for  $\xi, \eta$  as functions of  $t$  other functions  $\xi_1, \eta_1$ . Consequently it follows that *the value of the above quotient depends only upon the nature of the curve  $x=x(t), y=y(t)$ .*

184. We might generalize the problem treated above by requiring the curve  $x=x(t)$ ,  $y=y(t)$  which minimizes or maximizes the integral

$$I^{(0)} = \int_{t_0}^{t_1} F^{(0)}(x, y, x', y') dt,$$

while at the same time the following integrals have a prescribed value:

$$I^{(1)} = \int_{t_0}^{t_1} F^{(1)}(x, y, x', y') dt,$$

$$I^{(2)} = \int_{t_0}^{t_1} F^{(2)}(x, y, x', y') dt,$$

. . . . .

$$I^{(\mu)} = \int_{t_0}^{t_1} F^{(\mu)}(x, y, x', y') dt,$$

the functions  $F^{(0)}, F^{(1)}, \dots, F^{(\mu)}$  being of the same nature as the function  $F$  defined in Chapter I.

We must now consider the deformation of the curve caused by the variations

$$\bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1 + \epsilon_2 \xi_2 + \dots + \epsilon_\mu \xi_\mu,$$

$$\bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1 + \epsilon_2 \eta_2 + \dots + \epsilon_\mu \eta_\mu.$$

We have, then, if we write  $w_i = y' \xi_i - x' \eta_i$  ( $i=1, 2, \dots, \mu$ ), and suppose that the  $\xi$ 's and  $\eta$ 's vanish for  $t=t_0$  and  $t=t_1$ ,

$$\begin{aligned} \Delta I^{(0)} = & \epsilon \int_{t_0}^{t_1} G^{(0)} w dt + \epsilon_1 \int_{t_0}^{t_1} G^{(0)} w_1 dt + \dots \\ & + \epsilon_\mu \int_{t_0}^{t_1} G^{(0)} w_\mu dt + (\epsilon, \epsilon_1, \dots, \epsilon_\mu)_2, \end{aligned}$$



In order that the integral  $I^{(1)}$  have a maximum or a minimum value, it is therefore necessary that

$$D=0.$$

This determinant, when expanded, may be written in the form

$$\int_{t_0}^{t_1} [\lambda_0 G^{(0)} + \lambda_1 G^{(1)} + \dots + \lambda_\mu G^{(\mu)}] w dt = 0,$$

where  $\lambda_1$  is the first minor of  $\int_{t_0}^{t_1} G^{(1)} w dt$  in the determinant  $D$ .

Hence, as before (cf. Art. 79, where we had  $G=0$ ), we have here

$$\lambda_0 G^{(0)} + \lambda_1 G^{(1)} + \dots + \lambda_\mu G^{(\mu)} = 0.$$

185. Similarly, if in Art. 183 we denote the quotient  $\frac{W_{10}^{(0)}}{W_{10}^{(1)}}$  by  $\lambda$  and then give to  $W_{01}^{(0)}$  and  $W_{01}^{(1)}$  their values, we have

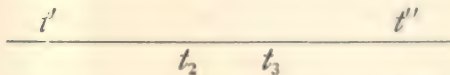
$$\int_{t_1}^{t_0} (G^{(0)} - \lambda G^{(1)}) w dt = 0.$$

From this it follows that

$$G^{(0)} - \lambda G^{(1)} = 0.$$

We may prove a very important theorem regarding the constant  $\lambda$ , viz:—*it has one and the same value for the whole curve; i. e., we always have the same value of  $\lambda$ , whatever part of the curve  $x=x(t)$ ,  $y=y(t)$  we may vary.* Consider the values of  $t$  laid off on a straight line, and suppose that the constant  $\lambda$  has a definite value for, say, the interval  $t_2 \dots t_3$  which also corresponds to a certain portion of curve. This value (see Art. 183) is independent of the manner in which

the portion of curve  $t_2 \dots t_3$  has been varied. Next consider an interval  $t' \dots t''$  which includes the interval  $t_2 \dots t_3$ ; then, there belongs to all the possible variations of the interval  $t' \dots t''$ , also



that variation by which  $t' \dots t_2$ , and  $t_3 \dots t''$  remain unchanged and only  $t_2 \dots t_3$  varies. As  $\lambda$  has a definite value for this interval and is independent of the manner in which the curve has been varied, it must have the same value for  $t' \dots t''$ .

186. The differential equation  $G^{(0)} - \lambda G^{(1)} = 0$  is the same as the one we would have if we require that the integral

$$\int_{t_0}^{t_1} F(x, y, x', y') dt$$

have a maximum or a minimum value, where  $F$  is written for the function

$$F^{(0)} - \lambda F^{(1)}.$$

Through this differential equation (See Art. 90)  $x$  and  $y$  are expressible in terms of  $t$  and  $\lambda$  and two constants of integration  $\alpha$  and  $\beta$  in the form

$$x = \phi(t, \alpha, \beta, \lambda),$$

$$y = \psi(t, \alpha, \beta, \lambda).$$

The curve represented by these equations is a solution of the problem, when indeed a solution is possible.

187. We prove next a very important theorem which often gives a criterion whether a sudden change in direction can take place or not within a stretch where the variation is unrestricted (cf. Art. 97). Suppose that on a position  $t = t'$ , where the variation is unrestricted, a sudden change in direction is experienced. On either side of  $t'$  take two points  $t_1$  and  $t_2$  so near to  $t'$  that within the intervals  $t_1 \dots t'$  and  $t' \dots t_2$  a similar discontinuity in change of direction is not had. Among the possible variations there is one such that the whole curve remains unchanged except the interval  $t_1 \dots t_2$ , which is, of course, varied in such a way that the integral  $I^{(1)}$  retains its value. The variation of the integral  $I^{(0)}$  depends then only upon the variation of the sum of integrals

$$\int_{t_1}^{t'} F^{(0)}(x, y, x', y') dt + \int_{t'}^{t_2} F^{(0)}(x, y, x', y') dt.$$

We cause a variation in the stretch  $t_1 \dots t_2$  by writing

$$\bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1,$$

$$\bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1,$$

where we assume that

$$(A) \quad \begin{cases} \xi, \xi_1, \eta, \eta_1 \text{ are all zero for } t=t_1 \text{ and } t=t_2, \\ \xi, \xi_1, \eta_1 \text{ are zero for } t=t_1, \\ \eta \neq 0 \text{ for } t=t_1. \end{cases}$$

We may then always determine  $\epsilon_1$  as a power-series in  $\epsilon$  so that  $\Delta f^{(1)} = 0$ .

If by  $\phi$  we denote an expression of the form  $\phi^{(0)} - \lambda \phi^{(1)}$ , we have (Art. 79)

$$\begin{aligned} \Delta f^{(0)} = & \epsilon \int_{t_1}^{t'} G w dt + \epsilon \int_{t'}^{t_2} G w dt + \epsilon \left[ \left( \xi - \lambda \xi_1 \right) \frac{\partial F}{\partial x'} + \left( \eta - \lambda \eta_1 \right) \frac{\partial F}{\partial y'} \right]_{t_1}^{t'} \\ & + \epsilon \left[ \left( \xi - \lambda \xi_1 \right) \frac{\partial F}{\partial x'} + \left( \eta - \lambda \eta_1 \right) \frac{\partial F}{\partial y'} \right]_{t'}^{t_2} + \epsilon(\epsilon). \end{aligned}$$

If the curve  $x = x(t)$ ,  $y = y(t)$  minimizes or maximizes the integral  $I^{(0)}$ , it is necessary that the coefficient of  $\epsilon$  on the right-hand side of the above expression be zero. Since  $G = 0$  for unrestricted variation, it follows from the assumption (A) that

$$\eta_{t'} \left[ \left( \frac{\partial F}{\partial y'} \right)_{t'}^- - \left( \frac{\partial F}{\partial y'} \right)_{t'}^+ \right] = 0.$$

If in the assumptions (A) we assume for  $t=t_1$  that  $\eta = 0$  and  $\xi \neq 0$ , we have an analogous equation for  $x'$ .

It therefore follows (cf. Art. 97) that

$$\begin{aligned} \left[ \frac{\partial (F^{(0)} - \lambda F^{(1)})}{\partial x'} \right]_{t'}^- &= \left[ \frac{\partial (F^{(0)} - \lambda F^{(1)})}{\partial x'} \right]_{t'}^+, \\ \left[ \frac{\partial (F^{(0)} - \lambda F^{(1)})}{\partial y'} \right]_{t'}^- &= \left[ \frac{\partial (F^{(0)} - \lambda F^{(1)})}{\partial y'} \right]_{t'}^+. \end{aligned}$$

We have then the theorem : *Along those positions which are free to vary of the curve which satisfies the differential equation  $G=0$ , the quantities  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  vary everywhere in a continuous manner, even on such positions of the curve where a sudden change in its direction takes place.*

188. It is obvious that these discontinuities may all be avoided, if we assume that  $\xi, \eta, \xi_1, \eta_1$  vanish at such points. This we may suppose has been done. We may also impose many other restrictions upon the curve ; for example, that it is to go through certain fixed points, or that it is to contain certain given portions of curve, or that it is to pass through a certain limited region. In all these cases there are points on the curve which cannot vary in a free manner. But whatever condition may be imposed upon the curve, the following theorem is true.

*All points which are free to vary (and there always exist such points) must satisfy the differential equation  $G^{(0)} - \lambda G^{(1)} = 0$ , and for all such points the constant  $\lambda$  has the same value.*

189. *The second variation.* We assume that the variations at the limits and at all points of the curve where there is a discontinuity in the direction, vanish. We also suppose that the variations  $\bar{\xi}, \bar{\eta}$  have been so chosen that  $\Delta I^{(1)} = 0$ .

We then have (cf. Art. 115):

$$\Delta I^{(0)} = \epsilon \delta I^{(0)} + \frac{\epsilon^2}{2} \int_{t_0}^{t_1} \left[ F_1^{(0)} \left( \frac{dw}{dt} \right)^2 + F_2^{(0)} w^2 \right] dt + (\epsilon)_3,$$

$$0 = \epsilon \delta I^{(1)} + \frac{\epsilon^2}{2} \int_{t_0}^{t_1} \left[ F_1^{(1)} \left( \frac{dw}{dt} \right)^2 + F_2^{(1)} w^2 \right] dt + (\epsilon)_3,$$

and consequently

$$\Delta I^{(0)} = \epsilon [\delta I^{(0)} - \lambda \delta I^{(1)}] + \frac{\epsilon^2}{2} \int_{t_0}^{t_1} \left[ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right] dt + (\epsilon)_3.$$

Since

$$\delta I^{(0)} - \lambda \delta I^{(1)} = \int_{t_0}^{t_1} (G^{(0)} - \lambda G^{(1)}) w \, dt = 0,$$

it follows that

$$\Delta I^0 = \frac{1}{2} \epsilon^2 \int_{t_0}^{t_1} \left[ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right] dt + (\epsilon)_3.$$

This last integral may be written at once (Art. 119) in the form

$$\Delta I^0 = \frac{\epsilon^2}{2} \int_{t_0}^{t_1} F_1 \left( \frac{dw}{dt} - \frac{w}{u} \frac{du}{dt} \right)^2 dt,$$

where  $u$  is determined from the differential equation (Art. 118)

$$J = F_1 \frac{d^2 u}{dt^2} + \frac{dF_1}{dt} \frac{du}{dt} - F_2 u = 0.$$

It follows here as a necessary condition for the existence of a maximum or a minimum that  $F_1$  for all portions of the curve at which there is free variation, must in the first case be everywhere *negative* and in the second case everywhere *positive* and must also be different from 0 and  $\infty$ . In order that this transformation of the integral be possible the equation  $J=0$  must admit of being integrated in such a way that  $u$  is different from zero on all portions of curve, which vary freely (Art. 128).

We shall determine in Chapter XVII whether the three necessary conditions thus formulated are also sufficient for a maximum or a minimum value of the integral  $I^u$ . By means of the example in the next Chapter, we shall also show that if there exists a curve, for which the first integral has a maximum or a minimum value while the second integral retains a given value, then the curve is determined through the three conditions, which are the same here as those formulated in Art. 135. The behavior of the  $\mathfrak{E}$ -function is then decisive regarding whether there in reality exists a maximum or a minimum.

## CHAPTER XIV.

## THE ISOPERIMETRICAL PROBLEM.

190. The isoperimetrical problem may be briefly stated as follows :

*Determine the curve of given length which maximizes or minimizes a certain definite integral.*

For example, it may be asked : *Among all curves of a given length joining two points, what is the form of the one which produces a minimum surface of revolution about a definite axis; or, along what arc of given length joining two fixed points does a particle under the influence of gravity descend in the shortest time?*

We shall consider here the Problem V of Chapter I, which may be again stated as follows : *Suppose that any portion of the plane is bounded in such a way that one can go from any point in it to any other point without crossing the boundaries. In this portion of plane a line returning into itself is to be so constructed that having a given length it incloses the greatest possible surface-area.*

Let  $x$  and  $y$  be such functions of  $t$  that for two definite values  $t_0$  and  $t_1$  the corresponding points fall together, and that while  $t$  goes from the smaller value  $t_0$  to the greater value  $t_1$ , the point  $x, y$  traverses in a positive direction the whole curve from the initial point to the end-point.

The surface-area, inclosed by the curve, is expressed by the integral

$$1) \ I^{(0)} = \frac{1}{2} \int_{t_0}^{t_1} (x y' - y x') dt,$$

and its perimeter by

$$2) \ I^{(1)} = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2} dt.$$

The problem proposed consists in expressing  $x$  and  $y$  as functions of  $t$  in such a manner that the first integral shall have the greatest possible value, while at the same time the second integral retains a given value.

It makes no difference where the origin of coördinates has been chosen; for by a transformation of the origin the second integral remains unchanged while the first integral is changed only by a constant. This does not alter the maximum property of the integral.

One may also add other conditions; for example: *That the curve go through a certain number of fixed points in a given order, or that it is to include certain portions of curve in a given order, etc.* The curve will then contain portions along which the variation is not free.

191. The function  $F$  is here

$$F = \frac{1}{2} (x y' - x' y) - \lambda \sqrt{x'^2 + y'^2}.$$

Instead of this function we may substitute another, since

$$\frac{d(xy)}{dt} = x y' + y x',$$

and consequently.

$$\frac{1}{2} (x y' - x' y) = \frac{1}{2} \frac{d}{dt} (xy) - y x'.$$

Now, if we integrate between the limits  $t_0 \dots t_1$ , the first term of the right-hand side of the above equation vanishes, since the end-point and the initial-point of the curve coincide. It follows, then, that

$$\frac{1}{2} \int_{t_0}^{t_1} (x y' - y x') dt = - \int_{t_0}^{t_1} y x' dt.$$

We may consequently give the function  $F$  the value

$$3) \quad F = -x' y - \lambda \sqrt{x'^2 + y'^2}.$$

From this we have

$$\frac{\partial F}{\partial x'} = -y - \frac{\lambda x'}{\sqrt{x'^2 + y'^2}}, \quad \frac{\partial F}{\partial y'} = -\frac{\lambda y'}{\sqrt{x'^2 + y'^2}}.$$

But since (Art. 187)  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  vary in a continuous manner along the portions of curve that vary freely, since also  $\lambda$  has the same constant value for the whole curve (Art. 185), and since the quantities that are multiplied by  $\lambda$  are nothing other than the direction-cosines of the tangent to the curve, it follows that the curve at every point, where the variation is free, changes its direction in a continuous manner.

192. The function  $F_1$  has the value

$$4) \quad F_1 = \frac{-\lambda}{(\sqrt{x'^2 + y'^2})^3}.$$

It is evident that  $F_1$  does not change sign, and since a maximum is to enter and consequently  $F_1$  is to be continuously negative, it follows that  $\lambda$  must be a positive constant.

193. In order to find the curve itself, we have to integrate the differential equation  $G^{(0)} - \lambda G^{(1)} = 0$ . This equation is equivalent (Art. 79) to the two equations

$$\frac{d}{dt} \frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0, \quad \frac{d}{dt} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0.$$

Since  $F$  does not contain  $x$  explicitly, the first of these equations gives

$$5) \quad \frac{\partial F}{\partial x'} = \text{const.}, \quad \text{or} \quad y + \frac{\lambda x'}{\sqrt{x'^2 + y'^2}} = b,$$

where  $b$  is an arbitrary constant. Since  $\frac{\partial F}{\partial x'}$  varies in a continuous manner for a portion of curve where there is free variation, it follows that the constant  $b$  retains the same value throughout such a portion of curve. The curve may, however, consist of separate portions which are free to vary, and for these the constant  $b$  may have different values.

If we take as the independent variable the arcs of curve measured from the origin, we have from 5),

$$6) \quad \frac{dx}{ds} = -\frac{1}{\lambda}(y-b),$$

and consequently, since  $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$ , it follows that

$$\left(\frac{dy}{ds}\right)^2 = 1 - \frac{1}{\lambda^2}(y-b)^2,$$

and

$$\frac{d^2y}{ds^2} = -\frac{1}{\lambda^2}(y-b) = \frac{1}{\lambda} \frac{dx}{ds}.$$

It is seen at once, if we integrate the last equation, that

$$7) \quad \frac{dy}{ds} = \frac{1}{\lambda}(x-a),$$

where  $a$  is an arbitrary constant; and consequently the equation of the curve is

$$8) \quad (x-a)^2 + (y-b)^2 = \lambda^2.$$

From the nature of the curve it is evident that  $\lambda$  is a positive constant.

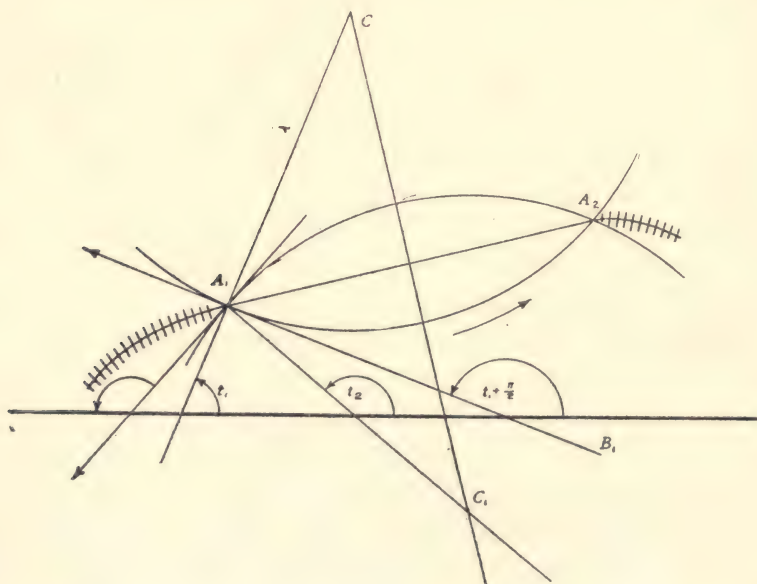
194. An immediate consequence is the theorem of Steiner, *that those portions of the curve, which are free to vary, must be the arcs of equal circles.* These circles may have different centers, since  $a$  and  $b$  are not determined. Each such arc of the circle may, however, lie on different sides of the chord joining two end-points; we have, therefore, to ascertain which of the two arcs is the one required.

The solutions of the differential equation are

$$x-a=\lambda \cos \frac{S-S_0}{\lambda}=\lambda \cos t,$$

$$y-b=\lambda \sin \frac{S-S_0}{\lambda}=\lambda \sin t,$$

as is seen from equations 6) and 7), when differentiated. Since  $\lambda$  is positive,  $s$  increases with  $t$  and since with increasing  $t$  the curve is traversed in the positive direction, we must take that arc for which this is also true. Let  $C$  be the center of the circle,  $A_1$  the



initial-point, and  $A_2$  the end-point of the arc. That arc will be the right one which lies on the positive side of  $CA_1$ , that is, on the side of the increasing  $t$ 's. For if  $t_1$  is the angle which the radius

$CA_1$  makes with the  $X$ -axis, and if  $x_1, y_1$  are the coördinates of the point  $A_1$ , then we have

$$\cos t_1 = \frac{1}{\lambda}(x_1 - a),$$

$$\sin t_1 = \frac{1}{\lambda}(y_1 - b),$$

and further the angle, which the tangent  $A_1 B_1$  drawn to the arc at the point  $A_1$  includes with the  $X$ -axis is  $t_1 + \frac{\pi}{2}$ . Consequently we have

$$\cos\left(t_1 + \frac{\pi}{2}\right) = -\sin t_1 = -\frac{1}{\lambda}(y_1 - b) = \left(\frac{dx}{ds}\right)_1,$$

$$\sin\left(t_1 + \frac{\pi}{2}\right) = \cos t_1 = \frac{1}{\lambda}(x_1 - a) = \left(\frac{dy}{ds}\right)_1,$$

formulae, which have the right signs. This would not be true if we took the other arc and also the tangent which is drawn in the other direction. Hence that arc is always to be taken, which, looking out from the center, is traversed in the *positive* direction.

195. If no conditions are imposed upon the curve and it is required to find among all isoperimetrical lines that one which offers the greatest surface-area, then the question is not of an *absolute maximum*, since the curve may be shoved anywhere in the plane without an alteration in its shape. The problem may be stated more accurately by saying that *the integral which represents the surface-area is not to admit of a positive increment, when all possible variations are introduced*. The problem thus formulated leads to exactly the same necessary conditions as before, namely that the first variation is to vanish, and consequently we have the same differential equation to solve. We have also the same condition for  $\lambda$ . Since the second variation can never be positive, and consequently  $F_1$  can not change its sign, we conclude as above that  $\lambda$  is positive. Since the whole curve is free to vary and since  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  are continuous functions for the whole trace, the constants  $a$  and  $b$  are the same for the whole curve; however, they remain undetermined. We have, consequently, the following result:

*If there exists a closed curve which with a given periphery includes the greatest surface-area, this curve is a circle.*

196. However, it has not as yet been proved that this property belongs to the circle. The treatment of the second variation is not sufficient, since only such variations have been employed where the distance between two corresponding points, and also the difference in direction at these points do not exceed certain limits.

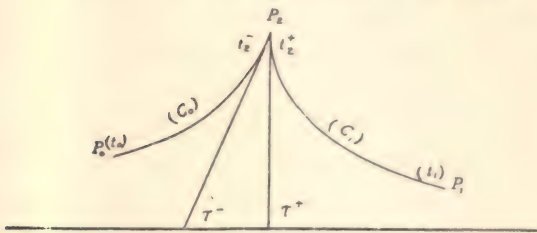
The further proof has to be made that every other curve forms the boundary of a smaller surface-area. The proof that the circle has this maximum property, ( a proof which is omitted in all previous solutions of the problem), has been considered so difficult that its solution has been denied to be in the province of the Calculus of Variations. We shall, however, in the next Chapter show that in the theorems already treated a means of overcoming this difficulty is offered. It will be seen that without the use of the second variation the desired result is reached in all cases where the function  $F_1$  does not change sign, not only at any point of the curve but also for any direction at any point.

## CHAPTER XV.

## RESTRICTED VARIATIONS. THE THEOREMS OF STEINER.

197. We shall consider in this Chapter some special cases of restricted variations. Suppose first that the path of integration is taken over two traces  $P_0 P_2$  and  $P_2 P_1$ .

We have for the first variation of the integral (Art. 79)



$$\delta I = \int_{t_0}^{t_2^-} G w dt + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t_0}^{t_2^-} + \int_{t_2^+}^{t_1} G w dt + \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t_2^+}^{t_1}.$$

Since the variation along the traces  $(C_0)$  and  $(C_1)$  is free, it follows that  $G=0$  for them, and consequently

$$\delta I = \left[ \frac{\partial F}{\partial x'} \xi + \frac{\partial F}{\partial y'} \eta \right]_{t_2^+}^{t_2^-}.$$

In order then for the first variation to be zero, it is necessary that

$$\left[ \frac{\partial F^-}{\partial x'} - \frac{\partial F^+}{\partial x'} \right] \xi + \left[ \frac{\partial F^-}{\partial y'} - \frac{\partial F^+}{\partial y'} \right] \eta = 0.$$

198. If *first* the conditions of the problem leave  $P_2$  free to vary in any direction, we must have, since  $\xi$  and  $\eta$  are arbitrary,

$$\frac{\partial F^-}{\partial x'} = \frac{\partial F^+}{\partial x'} \quad \text{and} \quad \frac{\partial F^-}{\partial y'} = \frac{\partial F^+}{\partial y'},$$

or, the curve consists of a single trace.



From this it follows, unless  $f(x, y) = 0$  at the point  $P_2$ , that

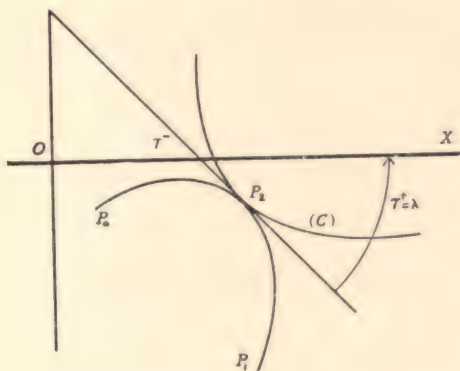
$$\cos(\tau - \lambda)^- = \cos(\tau - \lambda)^+,$$

or

$$(\tau - \lambda)^- = \pm(\tau - \lambda)^+ [\text{mod } \pi].$$

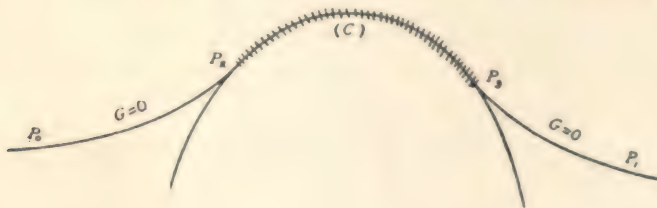
It is seen that the tangents to the two traces  $P_0 P_2$  and  $P_2 P_1$  at the point  $P_2$  have *either* one and the same tangent at  $P_2$  and are parts of one and the same curve, so that this case is the same as if  $P_2$  were not constrained, *or* they make with the tangent to the fixed curve equal angles  $T_1 P_1 M_1$  and  $T_2 P_2 M_2$ .

A limiting case is where  $\tau = \lambda$ , when again  $P_0 P_2$  and  $P_2 P_1$  form a continuous curve touching the fixed curve at the point  $P_2$ . The function  $\mathfrak{E}$  is here



$$\begin{aligned} \mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) &= f(x, y) \left[ 1 - \frac{\bar{x}'}{\sqrt{x'^2 + y'^2}} \frac{x'}{\sqrt{x'^2 + y'^2}} - \frac{\bar{y}'}{\sqrt{x'^2 + y'^2}} \frac{y'}{\sqrt{x'^2 + y'^2}} \right] \\ &= f(x, y) [1 - \cos(\tau - \lambda)] = 0, \text{ when } \tau = \lambda. \end{aligned}$$

200. Suppose that the path of integration coincides in part with one or more fixed curves, for example, with the curve (C).



Then we cannot say that  $G=0$  for the path of integration from  $P_2$  to  $P_3$ , but from the expression

$$\delta I = \int_{t_2}^{t_3} G w dt$$

it is evident that for the possibility of a *maximum*,  $w$  and  $G$  must have opposite signs, and the same signs for the possibility of a *minimum*.

201. In the general case, we made the substitutions

$$\begin{aligned} x &\parallel x + \epsilon \xi + \epsilon_1 \xi_1 + \epsilon_2 \xi_2 + \dots + \epsilon_\mu \xi_\mu, \\ y &\parallel y + \epsilon \eta + \epsilon_1 \eta_1 + \epsilon_2 \eta_2 + \dots + \epsilon_\mu \eta_\mu. \end{aligned}$$

Suppose that some point  $P$  of the path of integration is constrained to remain on a fixed curve, and, for simplicity, suppose that

$$\text{for } t=t'; \xi_1=\xi_2=\dots=\xi_\mu=0,$$

$$\text{and } \eta_1=\eta_2=\dots=\eta_\mu=0,$$

$$\text{but } \xi \neq 0 \text{ and } \eta \neq 0 \text{ at } t'.$$

Our previous equations (Art. 184) become now

$$\begin{aligned} \Delta I^{(1)} = \epsilon &\left[ \frac{\partial F^{(0)}}{\partial x'} \xi + \frac{\partial F^{(0)}}{\partial y'} \eta \right]_+^- + \epsilon \int_{t_0}^{t_1} G^{(0)} w dt \\ &+ \epsilon_1 \int_{t_0}^{t_1} G^{(0)} w_1 dt + \dots + \epsilon_\mu \int_{t_0}^{t_1} G^{(0)} w_\mu dt + (\epsilon^2), \end{aligned}$$

$$\begin{aligned} 0 = \Delta I^{(1)} = \epsilon &\left[ \frac{\partial F^{(1)}}{\partial x'} \xi + \frac{\partial F^{(1)}}{\partial y'} \eta \right]_+^- + \epsilon \int_{t_0}^{t_1} G^{(1)} w dt \\ &+ \epsilon_1 \int_{t_0}^{t_1} G^{(1)} w_1 dt + \dots + \epsilon_\mu \int_{t_0}^{t_1} G^{(1)} w_\mu dt + (\epsilon^2), \end{aligned}$$

$$\begin{aligned}
 o = \Delta I^{(2)} = & \epsilon \left[ \frac{\partial F^{(2)}}{\partial x'} \xi + \frac{\partial F^{(2)}}{\partial y'} \eta \right]_+^- + \epsilon \int_{t_0}^{t_1} G^{(2)} w dt \\
 & + \epsilon_1 \int_{t_0}^{t_1} G^{(2)} w_1 dt + \dots + \epsilon_\mu \int_{t_0}^{t_1} G^{(2)} w_\mu dt + (\epsilon^2), \\
 & \dots \dots \dots \\
 o = \Delta I^{(\mu)} = & \epsilon \left[ \frac{\partial F^{(\mu)}}{\partial x'} \xi + \frac{\partial F^{(\mu)}}{\partial y'} \eta \right]_+^- + \epsilon \int_{t_0}^{t_1} G^{(\mu)} w dt \\
 & + \epsilon_1 \int_{t_0}^{t_1} G^{(\mu)} w dt + \dots + \epsilon_\mu \int_{t_0}^{t_1} G^{(\mu)} w_\mu dt + (\epsilon_2).
 \end{aligned}$$

As in our previous discussion (Art. 184), it follows that

$$\begin{aligned}
 \delta I^{(0)} = & \left\{ \left[ \frac{\partial F^{(0)}}{\partial x'} \xi + \frac{\partial F^{(0)}}{\partial y'} \eta \right]_+^- + \int_{t_0}^{t_1} G^{(0)} w dt \right\} \lambda_0 \\
 & + \left\{ \left[ \frac{\partial F^{(1)}}{\partial x'} \xi + \frac{\partial F^{(1)}}{\partial y'} \eta \right]_+^- + \int_{t_0}^{t_1} G^{(1)} w dt \right\} \lambda_1 \\
 & \dots \dots \dots \\
 & + \left\{ \left[ \frac{\partial F^{(\mu)}}{\partial x'} \xi + \frac{\partial F^{(\mu)}}{\partial y'} \eta \right]_+^- + \int_{t_0}^{t_1} G^{(\mu)} w dt \right\} \lambda_\mu = o.
 \end{aligned}$$

Since further

$$\int_{t_0}^{t_1} \{ \lambda_0 G^{(0)} + \lambda_1 G^{(1)} + \dots + \lambda_\mu G^{(\mu)} \} w dt = o,$$

we have

$$0 = [\xi]_{\tau'} \left[ \lambda_0 \frac{\partial F^{(0)}}{\partial x'} + \lambda_1 \frac{\partial F^{(1)}}{\partial x'} + \dots + \lambda_\mu \frac{\partial F^{(\mu)}}{\partial x'} \right]_+ \\ + [\eta]_{\tau'} \left[ \lambda_0 \frac{\partial F^{(0)}}{\partial y'} + \lambda_1 \frac{\partial F^{(1)}}{\partial y'} + \dots + \lambda_\mu \frac{\partial F^{(\mu)}}{\partial y'} \right]_+.$$

If we write

$$F = \lambda_0 F^{(0)} + \lambda_1 F^{(1)} + \dots + \lambda_\mu F^{(\mu)},$$

and denote by  $\tau'$ , the angle which the tangent to the fixed curve at the point  $P'$  makes with the  $X$ -axis, the above expression becomes

$$\left[ \frac{\partial F}{\partial x'} \cos \tau' + \frac{\partial F}{\partial y'} \sin \tau' \right]_+ = 0.$$

If the point  $P'$  were not restricted, then  $\xi$  and  $\eta$  would be arbitrary, and we would have here

$$\left[ \frac{\partial F}{\partial x'} \right]_+ = 0 \text{ and } \left[ \frac{\partial F}{\partial y'} \right]_+ = 0,$$

which results compare with those of Art. 199.

202. We saw in the previous Chapter, if there existed a closed curve which with a given length bounded a maximum surface-area, that this curve was a circle. We supposed that it was possible for the circle to be situated entirely within the boundary of a given region. Suppose that this is not the case. The curve must then at least touch the given boundaries in two points or have a portion of the boundary in common. For we saw that the curve consisted of arcs of equal radii, and if these arcs did not touch the boundaries, there would necessarily be discontinuous changes in the direction of the variable curve. At such places, however, the surface-area could be increased without changing the perimeter.

203. Regarding the nature of the curve when it touches the boundaries, Steiner has given the two following theorems:

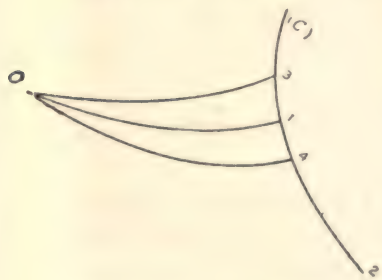
1) *If the curve coincides with a portion of the boundary, then the free portions of this curve are arcs of circles of equal radii, which are tangent to the boundary at the points of contact.*

2) *If the curve touches the boundary of the region in a point, then both parts of the curve are arcs of circles of equal radii, and the tangents to these two arcs at the point of contact with the boundary make with the tangent to the boundary at this point, equal angles.*

Steiner proved these theorems in a synthetic manner, and remarked that a synthetic-geometrical treatment seemed necessary, because the principles of the Calculus of Variations were not sufficient. Such remarks were, in a measure, justifiable, since up to that time only curves had been considered which satisfied the differential equation throughout their whole extent, and, therefore, no analytical means were known for the treatment of curves which in part coincided with given curves. However, there was no reason for saying that a method for the treatment of such problems was not within the province of the Calculus of Variations.

204. We shall show that the principles of the Calculus of Variations are sufficient to establish Steiner's theorems by proving two theorems due to Weierstrass, which are more general than the theorems of Steiner, and which have reference to the behavior of a curve at the points where it touches the boundary. The two theorems of Steiner are special cases of these theorems.

Suppose that the curve which satisfies the differential equation approaches the boundary at the point 1 and coincides with it up to the point 2. On the part of the curve which is traversed before we come to the boundary at 1, we take a point 0 so near to 1 that between 0 and 1 there is no sudden change in the direction of the curve.



The portion of curve 0 1 2 shall be so varied that we come to the boundary along another path from 0 to a point 3 before 1 or from 0 to a point 4 after 1 and then traverse the boundary to 2.

205. As we have already seen (Art. 161) the variation thereby produced in the integrals  $I^{(0)}$  and  $I^{(1)}$  may be expressed as follows: Let  $p_1, q_1$  be the direction-cosines of the curve 01 at the

point 1;  $\bar{p}_1, \bar{q}_1$  the direction-cosines of the boundary at this point;  $x_1, y_1$  the coördinates of the point 1, and  $\sigma$  the element of length of the boundary. Then we have, if the boundary is approached before the point 1 [see formula 5) Art. 161]

$$1) \begin{cases} \Delta I^{(0)} = \mathfrak{E}^{(0)}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)\sigma + \int_{t_0}^{t_1} G^{(0)} w dt + \left(\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}\right)_2, \\ \Delta I^{(1)} = \mathfrak{E}^{(1)}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)\sigma + \int_{t_0}^{t_1} G^{(1)} w dt + \left(\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}\right)_2; \end{cases}$$

and if the boundary is approached after the point 1 [see formula 6) Art. 161]

$$2) \begin{cases} \Delta I^{(0)} = -\mathfrak{E}^{(0)}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)\sigma + \int_{t_0}^{t_1} G^{(0)} w dt + \left(\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}\right)_2, \\ \Delta I^{(1)} = -\mathfrak{E}^{(1)}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)\sigma + \int_{t_0}^{t_1} G^{(1)} w dt + \left(\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}\right)_2. \end{cases}$$

Hence for case 1):  $\Delta I^{(0)} = \Delta I^{(0)} - \lambda \Delta I^{(1)} = (\mathfrak{E}^{(0)} - \lambda \mathfrak{E}^{(1)})\sigma + \left(\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}\right)_2$ ,

for case 2:  $\Delta I^{(0)} = \Delta I^{(0)} - \lambda \Delta I^{(1)} = -(\mathfrak{E}^{(0)} - \lambda \mathfrak{E}^{(1)})\sigma + \left(\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}\right)_2$ .

If the curve is to cause  $I^{(0)}$  to have a maximum or a minimum value while  $I^{(1)}$  remains unchanged, then (cf. Art. 189)  $\Delta I^{(0)}$  must have the same sign for both of the above variations. Hence, if the curve satisfies the differential equation  $G^{(0)} - \lambda G^{(1)} = 0$ , and if we write

$$\mathfrak{E}^{(0)} - \lambda \mathfrak{E}^{(1)} = \mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1),$$

then the function  $\mathfrak{E}$  must be zero at the point 1 of the boundary, because otherwise we could choose  $\sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}$  so small that the

sign of the whole expression depended upon the sign of the linear term, which in the first case is positive and in the second negative.

206. We saw (Art. 157) that

$$\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1) = (q_1 \bar{p}_1 - p_1 \bar{q}_1)^2 \int_0^1 F_1(x_1, y_1, p_k, q_k) (1-k) dk.$$

If  $\int_0^1 F_1(x_1, y_1, p_k, q_k) (1-k) dk$  is different from 0, (which must be determined in each separate case), it follows that

$$q_1 \bar{p}_1 - p_1 \bar{q}_1 = 0,$$

and therefore,

$$p_1 = \pm \bar{p}_1, \quad q_1 = \pm \bar{q}_1.$$

We wrote (Art. 157)

$$p_k = (1-k)p + k\bar{p},$$

$$q_k = (1-k)q + k\bar{q},$$

and consequently, if we take the lower sign, so that  $p_1 = -\bar{p}_1$ ,  $q_1 = -\bar{q}_1$ , then it may happen that  $F_1$  becomes infinitely large within the limits of integration, because for the value  $k = \frac{1}{2}$  both  $p_k$  and  $q_k$  are zero (see Art. 157).

In general, we have

$$p_1 = \bar{p}_1, \quad q_1 = \bar{q}_1 \quad (\text{cf. Art. 199}).$$

A special investigation must be made in the other case for every particular problem. We, therefore, have the theorem:

*If the curve which satisfies the differential equation approaches the boundary at a point and then coincides with a portion of the boundary, the direction at the point of contact can suffer no discontinuous change.*

The same result is derived in an analogous manner for the point where the curve leaves the boundary after having coincided with a portion of it.

207. We have tacitly assumed that there is no sudden change in the direction of the boundary at the point 1. But if this is the case and if  $\bar{p}_2, \bar{q}_2$  are the direction-cosines with which one approaches the point 1, and  $\bar{p}_1, \bar{q}_1$  those with which one leaves the point 1, then we have for  $\Delta I^{(0)}$  the expression:

$$\text{in the first case: } \Delta I^{(0)} = \mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_2, \bar{q}_2)\sigma + (\quad)_2;$$

$$\text{in the second case: } \Delta I^{(0)} = -\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)\sigma + (\quad)_2.$$

In a following Chapter (Art. 221), it will be proved that, if a maximum or a minimum is to appear, the function  $\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q})$  must have continuously the same sign for every point of the curve which is varied and for arbitrary directions  $p, q$  along it; in the first case this sign must *not* be *positive*, and in the second case it must *not* be *negative*.

From this it follows, for the case of both maximum and minimum, that we must again have

$$\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1) = 0,$$

while  $\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_2, \bar{q}_2)$  remains arbitrary.

For, if we are seeking a minimum, after the theorem just cited, the function  $\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)$  cannot be negative; but it cannot be positive because in virtue of the equation

$$\Delta I^{(0)} = -\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1)\sigma + (\quad)_2,$$

$I^{(0)}$  would for certain variations experience a negative change. Hence we must have:

$$\mathfrak{E}(x_1, y_1, p_1, q_1, \bar{p}_1, \bar{q}_1) = 0.$$

The same is true of the point where the curve leaves the boundary, so that we have the same results for the end-point as those just given for the initial-point. The results may be stated as follows:

*If the curve for which there is to appear a maximum or a minimum meets the boundary and traverses a portion of it, then at the point where it first comes to the boundary and at the point where it leaves the boundary, the two curves must be so situated that the tangents are the same for both curves. But if at these*

points there is a discontinuous change in direction of the boundary curve, then the direction of the curve as it approaches the boundary and that of the boundary at the point of approach may be quite arbitrary.

This is the *first* of Weierstrass' theorems.

208. We consider next the case where the curve meets the boundary in one point and then leaves it. Let 01 and 12 be the two portions of curve that satisfy the differential equation and meet the boundary at the point 1. Take the points 0 and 2 so near to 1 that within the intervals 01 and 12 there are no sudden changes in direction. We vary the curve 012 by going from the point 1 to a point 3 on the boundary. The point 3 is connected with the points 0 and 2 by curves which do not necessarily satisfy the differential equation, but are subject to the condition that the integral  $I^{(1)}$  remains unaltered by this variation.

Let  $p, q$  be the direction of 01 at 1,

$p_1, q_1$  the direction of 12 at 1,

and let the coördinates  $x, y$  which belong to the different points be indicated by the corresponding indices.

Then, as we have already seen (Arts. 79 and 154)

$$I_{03}^{(0)} - I_{01}^{(0)} = F^{(1)}(x_1, y_1, p, q)(x_3 - x_1) \\ + F^{(2)}(x_1, y_1, p, q)(y_3 - y_1) + \left( \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

$$I_{32}^{(0)} - I_{12}^{(0)} = -F^{(1)}(x_1, y_1, p_1, q_1)(x_3 - x_1) \\ - F^{(2)}(x_1, y_1, p_1, q_1)(y_3 - y_1) + \left( \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2;$$

and consequently

$$I_{032}^{(0)} - I_{012}^{(0)} = [F^{(1)}(x_1, y_1, p, q) - F^{(1)}(x_1, y_1, p_1, q_1)](x_3 - x_1) \\ + [F^{(2)}(x_1, y_1, p, q) - F^{(2)}(x_1, y_1, p_1, q_1)](y_3 - y_1) \\ + \left( \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2.$$



be either positive or negative; and since this quantity (neglecting a constant factor) is the distance from the dividing line, it is seen that it becomes infinitely small of the first order with  $\xi, \eta$ .

We may, therefore, choose  $\xi, \eta$  so small that

$$I_{032}^{(0)} - I_{012}^{(0)}$$

has the same sign as  $\mathfrak{E}\delta - \mathfrak{E}_1\delta_1$ , and, therefore, may be either positive or negative. Hence we must have

$$\mathfrak{E}\delta - \mathfrak{E}_1\delta_1 = 0.$$

Accordingly, the direction of the tangent to the boundary curve must coincide with that of the dividing line.

The sines of the angles which the dividing line makes with the  $\delta$  and  $\delta_1$ -axes, that is, with the tangents to the two portions of curve at the point where they meet the boundary, are to each other as  $\mathfrak{E}_1$  is to  $\mathfrak{E}$ , if the two angles are measured in opposite directions.

Weierstrass' *second* theorem may accordingly be stated as follows:

*If the curve which satisfies the differential equation meets the boundary in only one point and then leaves it, the tangents to the two portions of curve at this point, make with the tangent to the boundary at the same point, angles whose sines are to each other as  $\mathfrak{E}_1$  is to  $\mathfrak{E}$ .*

The second theorem of Steiner relative to the isoperimetrical problem is only a special case of this theorem. In this problem we have  $\mathfrak{E}_1 = \mathfrak{E}$  so that the two angles which the two tangents to the curves make with the tangent to the boundary curve are equal.

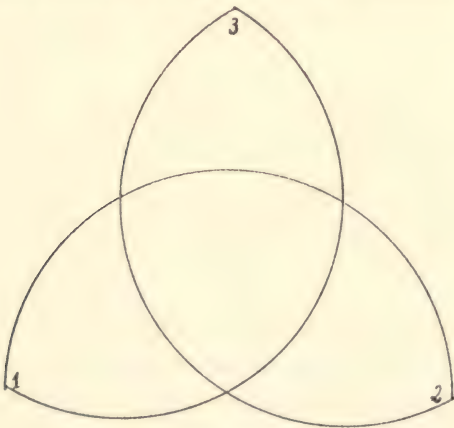
210. We have shown that the curve in every point where the variation is free satisfies one and the same differential equation and that the constant  $\lambda$  has the same value for the whole curve (Art. 185). This leads to a certain paradox: If we reverse the isoperimetrical problem and *seek the shortest line among all those lines which inclose a given surface-area, we come to the differential equation of the isoperimetrical problem.*

We have  $F = F^{(1)} - \lambda F^{(0)}$  in the place of  $F = F^{(0)} - \lambda F^{(1)}$  which occurred before; still on this account the nature of the differential equation is not changed, since there is only a change in the constants. It is, however, *à priori* clear that the solution of the two problems must be the same; for, if it were possible to keep the surface-area constant and shorten the perimeter, it is evident that with the original perimeter we could have inclosed a greater surface-area. Hence, the curve, which has been derived from the differential equation of the first problem, satisfies also the inverse problem. We consequently have as the solution of the second problem the theorem: *The curve, wherever there is free variation, consists of arcs of circles which have equal radii.*

211. PROBLEM. *Three points 1, 2, 3 not lying in the same straight line are given in the plane and it is required to draw a line through them in a definite order, which includes a given surface-area and at the same time has the shortest possible length.*

We know that a circle  $W$ , say, fulfils these requirements, if the given area is the same as that included by a circle, which is determined by the three points 1, 2, 3.

But if the surface-area is greater or smaller than  $W$ , then the arcs of circles must be drawn outward or inward. If, however, the area is very small, we cannot draw arcs of circles so as to in-



close this area without crossing one another, and we do not admit into consideration the areas that are described in the opposite directions.

The problem may be solved as follows: The curve, although not being limited by further conditions, need not vary everywhere in a free manner, and, consequently, it is not necessarily constituted out of

arcs of circles. For if we assume that the curve is not to cross itself, then of itself it may offer barriers which obstruct free variation.

If, for example, the curve 0 1 2 3 partially overlaps so that the portion 1 3 coincides up to the point 2 with the portion 0 1, then among all possible variations, there are present those where 0 1 remain unchanged and only 1 3 varies; and since the curve is not to cross itself, the variation of the portion 1 2 can take place only on the side of 0 1 on which the point 3 lies, and, consequently, the freedom of the variation of the curve is essentially limited.

In itself the requirement that the curve is not to cut itself is not necessary, as the integrals that appear have a meaning also for this case.

If there are overlapping portions of curve, then we may allow such variations to enter that points coincident before the variation may also coincide after the variation, without the second integral changing its value. We shall investigate the kind of differential equation that is thereby produced for these portions of curve.

212. The following investigation is also applicable to the case where the second integral is not present. We have simply to make  $\lambda=0$ .

We introduce the variations

$$\bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1,$$

$$\bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1.$$

It has been shown that the first variation of  $I^{(0)}$  is identical with

$$\delta \int (F^{(0)} - \lambda F^{(1)}) dt,$$

provided that  $\epsilon_1$  can be expressed as a power-series in  $\epsilon$  in such a way that the total variation of the second integral vanishes.

This  $\delta I^{(0)}$  can be brought to the form

$$\delta I^{(0)} = \epsilon \int G(y' \xi - x' \eta) dt.$$

In the former treatment  $\xi$  and  $\eta$  were entirely arbitrary, except that at certain points and along certain portions of curve they vanished. Wherever they were arbitrary it was necessary that  $G=0$ .

In the case before us we have in addition those portions of curve which overlap the curve several times without crossing it. The differential equation, which these portions of curve satisfy, may be obtained as follows:

We have, since  $dt$  is a positive increment of  $t$  (Art. 68),

$$F(x, y, x', y') dt = F(x, y, x' dt, y' dt) = F(x, y, dx, dy).$$

Let 1 2 be a portion of curve that is traversed several times. The integral over this portion of curve, after it has been traversed once from the point 1 to the point 2, may be written in the form

$$\int_{x_1, y_1}^{x_2, y_2} F(x, y, dx, dy) = \int_1^2 F(x, y, dx, dy).$$

The portion of the integral taken over the curve in the opposite direction is

$$\int_2^1 F(x, y, dx, dy).$$

If this portion of curve is traversed  $\mu$  times in the first direction and  $\nu$  times in the second, and if all the variations except those that relate to this portion of curve be put equal to zero, then the variation of the whole integral is equal to the variation of the sum of integrals:

$$\mu \int_1^2 F(x, y, dx, dy) + \nu \int_2^1 F(x, y, dx, dy).$$

But since

$$\int_2^1 F(x, y, dx, dy) = \int_1^2 F(x, y, -dx, -dy),$$

the above sum is equal to

$$\mu \int_1^2 F(x, y, dx, dy) + \nu \int_1^2 F(x, y, -dx, -dy);$$

or, if we put

$$\mu F(x, y, dx, dy) + \nu F(x, y, -dx, -dy) = \bar{F}(x, y, dx, dy),$$

the sum is

$$\int_1^2 \bar{F}(x, y, dx, dy).$$

The portion of curve 12 is traversed only once for this integral, and consequently the variations are quite free. The interval 12 must therefore satisfy the differential equation which is derived for the function  $\bar{F}(x, y, x', y')$  in the same manner as in the former investigations, where  $F(x, y, x', y')$  was the function considered.

213. If, for example, the problem is to *determine the curve which with a given surface-area has the shortest perimeter*, then

$$F(x, y, dx, dy) = \sqrt{dx^2 + dy^2} - \lambda y dx,$$

and for  $\mu = \nu$ ,

$$\begin{aligned} \bar{F}(x, y, dx, dy) &= \mu [\sqrt{dx^2 + dy^2} - \lambda y dx + \sqrt{dx^2 + dy^2} + \lambda y dx] \\ &= 2\mu \sqrt{dx^2 + dy^2}. \end{aligned}$$

Consequently the differential equation leads to a straight line.

But if  $\mu \leq \nu$ , we have

$$\bar{F} = (\mu + \nu) \sqrt{dx^2 + dy^2} - \lambda(\mu - \nu) y dx.$$

The corresponding differential equation is of the form

$$G^{(1)} - \lambda_1 G^{(0)} = 0,$$

where  $\lambda_1 = \lambda \frac{\mu - \nu}{\mu + \nu}$ ; it, therefore, leads to the arc of a circle which has a different radius than the one belonging to the portions of curve where the variation is free.

This case, however, does not in reality appear unless there are certain modifications; for, if we traverse such an arc of circle twice in opposite directions, the portion of surface-area thereby obtained is zero. We may, however, shorten the perimeter by taking instead of the arc of a circle the chord which joins its end-points, this being the first solution above. If, further, the same arc of circle was traversed several times, then in case there are

not special modifications, we may neglect the first two times or the first  $2n$  times that the arc is traversed (owing to which the perimeter is shortened) without changing the surface-area,

Taking also into consideration the case where  $\mu=\nu=1$ , when a straight line enters, we have to see which of these portions of curve (straight line or arc) can be used to form the required curve and how they are to be grouped. We have then to seek all possible kinds of combinations and make proof of their admissibility

We consider any configuration and cause it to vary. Since the nature of the curve is known and only the end-points of the individual portions are undetermined, we have to subject these to variations. The previous theorems are fully sufficient for carrying this out. We, therefore, have a means of determining whether such a configuration of the individual portions is, or is not possible.

Since the individual portions satisfy their differential equations, the first variations of the corresponding integrals will depend only upon the variation of the end-points; and, if we apply this to all the portions of the curve, we will have a linear function of all the variations of the coördinates of the individual end-points.

These end-points may be subjected to further restrictions; for example, they may be compelled to lie upon given curves, etc.

By the application of previously developed theorems, we have certain equations for the determination of the possible position of the end-points of the individual portions and we may thus see whether a definite configuration is, or is not possible.

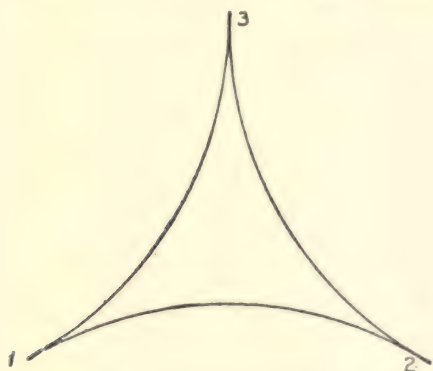
214. At all events, for the grouping which has been thus determined the first variation of the integral vanishes, but this does not of itself denote that a maximum or a minimum has appeared. This determination is a problem in the usual Theory of Maxima and Minima. Since, as soon as the individual portions of curve have been found, we can also determine the integrals for them whose values depend only upon the constants  $\lambda$  that have been introduced and the coördinates of the end-points. We have thus an ordinary function of a finite number of variables, and the question is whether

this function really satisfies the conditions of a maximum or a minimum. This subject is treated in the Theory of Maxima and Minima, involving several variables.

Thus we may at least determine whether or not a certain formation of the curve satisfies the problem. For example, a curve is required to pass in a definite order through the points 1, 2 and 3 and which having the smallest possible perimeter is to inscribe a given surface-area. The curve in question consists of three portions which pass through 1 and 2, 2 and 3, 3 and 1. These portions are the arcs of circles with equal radii, if the given surface-area is sufficiently large. This radius is to be determined from the given value of the surface.

The integral  $I^{(0)}$  is a function of the constants that appear, and it may be shown that this integral is in reality a minimum when the constants have been correctly determined.

But if the surface-area is not sufficiently large, then the portions of curve must partially overlap one another, and the portions along which this happens are straight lines. The curve cannot end in points which are perfectly free to vary; for if this were the case, we could so vary the point that the surface-area remained the same while its length became shorter. These points must lie along straight lines which pass through the three given points.



It is thus found that the curve consists in reality of three arcs of circles which are described with equal radii and which mutually touch one another and go off into straight lines that pass through the given points, as shown in the figure.

215. It is seen that the solution of the problem is independent of the position of the points 1, 2, 3 relative to one another; for we can slide the points 1, 2, 3 backward and forward upon the straight lines without causing the curve to lose the property of having the minimum length. It is essential only in what manner the points

are chosen where the straight lines come together with the arcs of the circles. These points corresponding to the points 1, 2, 3 may be denoted by  $1'$ ,  $2'$ ,  $3'$ . If the portion  $2'1'1$  be considered as a fixed boundary and the end-point of  $3'1'$  varies along it, it follows from a theorem already given (Art. 206), that  $3'1'$  must so touch the boundary, that the curve  $3'1'1$  does not change its direction abruptly. Hence every two arcs of circles must touch at the points where they come together. Since the radii of the arcs of circles are equal, it follows that the three centers of the arcs of circles form an equilateral triangle, and consequently the three arcs of circles are of equal length. Therefore every two straight lines form an angle of  $120^\circ$  with each other, and thus the solution of the problem is uniquely determined. The above problem was proposed by Todhunter in the Mathematical Tripos Examination of 1865. It is treated by him (Researches in Calculus of Variations, pp. 44 et seq.).

## CHAPTER XVI.

 THE DETERMINATION OF THE CURVE OF GIVEN LENGTH  
AND GIVEN END-POINTS, WHOSE CENTER OF  
GRAVITY LIES THE LOWEST.

216. To solve the problem of this Chapter, let the  $Y$ -axis be taken vertically with the positive direction upward, and denote by  $S$  the length of the whole curve. If the coördinates of the center of gravity are  $x_0, y_0$ , then  $y_0$  is determined from the equation

$$y_0 = S \int_{t_0}^{t_1} y \sqrt{x'^2 + y'^2} dt, \quad \text{where } S = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2} dt.$$

The problem is: *So determine  $x$  and  $y$  as functions of  $t$  that the first integral will be a minimum while the second integral retains a constant value.* (See Art. 16).

The property that the center of gravity is to lie as low as possible must also be satisfied for every portion of the curve; for if this were not true, then we could replace a portion of the curve by a portion of the same length but with a center of gravity that lies lower, with the result that the center of gravity of the whole curve could be shoved lower down, and consequently the original curve would not have the required minimal property.

We have here

$$F^{(0)} = y \sqrt{x'^2 + y'^2}, \quad F^{(1)} = \sqrt{x'^2 + y'^2},$$

$$F = (y - \lambda) \sqrt{x'^2 + y'^2};$$

and therefore

$$\frac{\partial F}{\partial x'} = \frac{x'(y-\lambda)}{\sqrt{x'^2+y'^2}}, \quad \frac{\partial^2 F}{\partial x' \partial y'} = \frac{-x'y'(y-\lambda)}{(\sqrt{x'^2+y'^2})^3}, \quad \frac{\partial F}{\partial y'} = \frac{y'(y-\lambda)}{\sqrt{x'^2+y'^2}},$$

$$F_1 = \frac{y-\lambda}{(\sqrt{x'^2+y'^2})^3}.$$

We exclude once for all the case where the two given points lie in the same vertical line, because then the integral for  $S$  does not express for every case the absolute length of the curve; for example, when a certain portion of the curve overlaps itself. Similarly we exclude the case where the given length  $S$  is exactly equal to the length between the two points on a straight line; for, in this case, the curve cannot be varied and at the same time retain the constant length.

217. Since  $F_1$  must be positive, a minimum being required, it follows that  $(y-\lambda) > 0$ . Since further,  $\frac{\partial F}{\partial x'}$  and  $\frac{\partial F}{\partial y'}$  vary in a continuous manner along the whole curve, and since these quantities differ from the direction-cosines only through the factor  $y-\lambda$ , which varies in a continuous manner, it follows that the curve changes everywhere its direction in a continuous manner.

The function  $F$  is the same as the function  $F$  which appeared in Art. 7, except that here we have  $y-\lambda$  instead of  $y$  in that problem. Since the differential equation here must be the same as in the problem just mentioned, we must have as the required curve

$$\begin{cases} x = a \pm \beta t, \\ y = \lambda + \frac{1}{2}\beta(e^t + e^{-t}), \end{cases}$$

the equation of a catenary.

Since  $y-\lambda > 0$ , it follows that  $\beta$  is a positive constant. For  $S$  we have the value

$$S = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2} dt = \frac{\beta}{2} [e^{t_1} - e^{-t_1} - (e^{t_0} - e^{-t_0})].$$

218. We have next to investigate whether and how often a catenary may be passed through two points and have the length  $S$ ; that is, whether and in how many different ways it is possible to determine the constants  $\alpha$ ,  $\beta$ ,  $\lambda$  in terms of  $S$  and the coördinates of the given points. If we denote the coördinates of these points by  $a_0$ ,  $b_0$ ,  $a_1$ ,  $b_1$ , then is

$$\begin{aligned} a_0 &= \alpha \mp \beta t_0, \quad a_1 = \alpha \mp \beta t_1, \\ b_0 &= \lambda + \frac{\beta}{2}(e^{t_0} + e^{-t_0}), \quad b_1 = \lambda + \frac{\beta}{2}(e^{t_1} + e^{-t_1}), \\ S &= \frac{\beta}{2} \left\{ (e^{t_1} - e^{-t_1}) - (e^{t_0} - e^{-t_0}) \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} a_1 - a_0 &= \pm \beta(t_1 - t_0), \\ b_1 - b_0 &= \frac{\beta}{2} [(e^{t_1} + e^{-t_1}) - (e^{t_0} + e^{-t_0})]. \end{aligned}$$

We have assumed that  $t_1 > t_0$ , and consequently we have to take the upper or lower sign according as  $a_1 - a_0 > 0$  or  $a_1 - a_0 < 0$ . It is clear that we may always take  $a_1 - a_0 > 0$ , since we may interchange the point  $a_1$ ,  $b_1$  with the point  $a_0$ ,  $b_0$ , and *vice versa*.

We shall accordingly take the upper sign. If we write

$$\frac{t_1 - t_0}{2} = \mu, \quad \frac{t_1 + t_0}{2} = \nu,$$

then  $\mu$  is a positive quantity and we have

$$\begin{aligned} a_1 - a_0 &= +2\mu\beta, \\ b_1 - b_0 &= \frac{\beta}{2} (e^\mu - e^{-\mu}) (e^\nu + e^{-\nu}), \\ S &= \frac{\beta}{2} (e^\mu - e^{-\mu}) (e^\nu + e^{-\nu}), \\ \frac{b_1 - b_0}{S} &= \frac{1 - e^{-2\nu}}{1 + e^{-2\nu}} = -\frac{1 - e^{2\nu}}{1 + e^{2\nu}}, \\ \frac{d}{d\nu} \left( \frac{b_1 - b_0}{S} \right) &= \frac{4}{(e^\nu + e^{-\nu})^2}. \end{aligned}$$

Since this derivative is continuously positive, the expression  $\frac{b_1-b_0}{S}$  varies in a continuous manner from  $-1$  to  $+1$ , while  $\nu$  increases from  $-\infty$  to  $+\infty$ . Hence for every real value of  $\nu$  there is one and only one real value of  $\frac{b_1-b_0}{S}$  which is situated between  $-1$  and  $+1$ , and *vice versa* to every value of  $\frac{b_1-b_0}{S}$  situated between  $-1$  and  $+1$  there is one and only one real value of  $\nu$ . Since we excluded the case where  $S$  was equal to the length along a straight line between the two given points, it follows that  $S$  is always greater than  $b_1-b_0$  and consequently  $\frac{b_1-b_0}{S}$  is in reality a proper fraction. Hence  $\nu$  is *uniquely* determined through  $\frac{b_1-b_0}{S}$ .

219. We have further

$$\frac{S}{a_1-a_0} = \frac{(e^\mu - e^{-\mu})}{2\mu} \cdot \frac{(e^\nu + e^{-\nu})}{2},$$

or

$$\frac{2\mu}{e^\mu - e^{-\mu}} = \frac{a_1 - a_0}{S \sqrt{1 - \left(\frac{b_1 - b_0}{S}\right)^2}} = \frac{a_1 - a_0}{\sqrt{S^2 - (b_1 - b_0)^2}}.$$

The right-hand side is a given positive quantity which we may denote by  $M$ . It is seen that

$$\frac{d}{d\mu} \left( \frac{2\mu}{e^\mu - e^{-\mu}} \right) = -2 \frac{[(\mu-1)e^\mu + (\mu+1)e^{-\mu}]}{(e^\mu - e^{-\mu})^2}.$$

By its definition  $\mu$  is always greater than  $o$ . If  $\mu$  is situated between  $1$  and  $\infty$ , the right-hand side of the equation is always negative. Since further the differential quotient of the expression  $(\mu-1)e^\mu + (\mu+1)e^{-\mu}$  is never less than  $o$  while  $\mu$  varies from  $o$  to  $1$ , it is seen that this expression increases continuously when  $\mu$  varies from  $o$  to  $1$ ; hence the differential quotient of  $\frac{2\mu}{e^\mu - e^{-\mu}}$  is continuously negative, and consequently

$$\frac{d}{d\mu} \left( \frac{2\mu}{e^\mu - e^{-\mu}} \right) < o \text{ for } o < \mu < \infty.$$

Consequently the expression  $\frac{2\mu}{e^\mu - e^{-\mu}}$ , or the quantity  $M$ , continuously decreases from 1 to 0 while  $\mu$  takes the values from 0 to  $\infty$ , and therefore to every value of  $M$  lying between 0 and 1 there is one and only one value of  $\mu$  situated between 0 and  $\infty$ .

Since by hypothesis  $M$  is always a positive proper fraction, it follows from the above that  $\mu$  is uniquely determined through the given quantities. Through  $\mu$  and  $\nu$  and the other given quantities we may also determine uniquely  $\alpha, \beta, \lambda$ ; and consequently if  $S$  is taken sufficiently large, it is possible to lay one and only one catenary between the given points which satisfies the given conditions.

If, then, there exists a curve which is a solution of the problem, this curve is a catenary. We have not yet proved that in reality for this curve the first integral is a minimum. The sufficient criteria for this will be developed in the next Chapter.

## CHAPTER XVII.

## THE SUFFICIENT CONDITIONS.

220. In a similar manner as in the case of free variation (see Art. 159) there is also a way of solving completely the general problem of restricted variation without making use of the second variation. Let the differential equation be found through the variation of the integrals. The required curve must necessarily satisfy this equation. Let the portion of curve under consideration be so limited that for every point of it  $F^{(0)}$  and  $F^{(1)}$  are regular functions in  $x, y, x', y'$  and for no point on it the function  $F_1$  becomes zero or infinite. The case where the portion of curve contains singular points will be left for a special investigation in each particular problem.

221. Let 0 and 1 be the end-points of the portion of curve in question. Through an arbitrary point 2 of this curve we draw any regular curve and on it take a point 3 so near to 2 that we may join 0 and 3 by a curve which satisfies the differential equation. The line 0 3 2 1 is a possible variation of 0 1. The change which an integral



$$I^{(0)} = \int_{t_0}^{t_1} F^{(0)}(x, y, x', y') dt$$

suffers through this variation, takes the form (Art. 205)

$$\Delta I^{(0)} = \mathfrak{E}^{(0)}(x_2, y_2, p_2, q_2, \bar{p}_2, \bar{q}_2) \sigma + \int_{t_0}^{t_1} G^{(0)} w dt + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

where  $\sigma$  is the length of 23 taken in the positive direction;  $p_2, q_2$  denote the direction-cosines of 02 at 2;  $\bar{p}_2, \bar{q}_2$ , those of 3'2 at 2 and  $x_2, y_2$  the coördinates of 2.

If we have two integrals, and if the variation is such that one of the two integrals remains unchanged, and if  $I^{(1)}, \mathfrak{E}^{(1)}, G^{(1)}$  denote the corresponding quantities for the second integral, then we have

$$\Delta I^{(0)} = \mathfrak{E}^{(0)} \sigma + \int_{t_0}^{t_1} G^{(0)} w dt + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

$$0 = \mathfrak{E}^{(1)} \sigma + \int_{t_0}^{t_1} G^{(1)} w dt + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2.$$

Hence if, as usual, we denote the quantity  $\mathfrak{E}^{(0)} - \lambda \mathfrak{E}^{(1)}$  by  $\mathfrak{E}$ , then is

$$\Delta I^{(0)} = \mathfrak{E} \sigma + \int_{t_0}^{t_1} (G^{(0)} - \lambda G^{(1)}) dt + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

and, if the curve satisfies the differential equation  $G^{(0)} - \lambda G^{(1)} = 0$ , it follows that

$$\Delta I^{(0)} = \mathfrak{E} \sigma + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2.$$

From this equation it is seen that the function  $\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q})$  along the whole portion of curve cannot have opposite signs for any two pairs of values  $\bar{p}, \bar{q}$ , as  $\Delta I^{(0)}$  must always have the same sign.

222. As in Art. 157, we may write  $\mathfrak{E}$  in the form

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = (q\bar{p} - p\bar{q})^2 \int_0^1 [F_1^{(0)}(x, y, p_k, q_k) - \lambda F_1^{(1)}(x, y, p_k, q_k)] (1-k) dk,$$

where  $p_k = (1-k)p + k\bar{p}$ ,  $q_k = (1-k)q + k\bar{q}$ .

It follows at once from the preceding Article that

$$F_1^{(0)}(x, y, p, q) - \lambda F_1^{(1)}(x, y, p, q)$$

cannot have values with different signs for any values of  $p, q$ .

*The converse, however, is not true.* (See Art. 160). The condition that  $\mathfrak{E}$  cannot change its sign in so far as every arbitrary direction  $\bar{p}, \bar{q}$  is concerned has a further significance.

For erect lines along the curve 01 perpendicular to the plane of this curve. On these perpendiculars take lengths equal in value to the second integral, where in each case the integration is taken from 0 to the foot of the perpendicular. Then to the curve 01 there corresponds a curve in space 01', where the points in space are marked by indices corresponding to the points in the plane.

Thus to every curve through the point 0 and lying in this plane there corresponds a curve in space. We say that a curve in space satisfies the differential equation of the problem if its projection satisfies the differential equation  $G^{(0)} - \lambda G^{(1)} = 0$ , although  $\lambda$  need not have the same value for all the curves.

223. Now suppose that we can envelop the curve 01' in space in the following manner: The point 0 is to lie on the boundary, and the point 1' within the space enveloped; further, it is to be possible to draw from 0 to every point within this enveloped space at least one curve which satisfies the differential equation; and, when such a curve has been drawn from 0 to any point  $P$  within the enveloped space, it must be possible to draw a curve between 0 and a point neighboring to  $P$  which also satisfies the differential equation. This curve must lie everywhere as near as we wish to the first curve, and the associated  $\lambda$ 's can differ from one another only by arbitrarily small quantities.

If the end-point describes a continuous curve in the enveloped space, then we may draw a series of curves, corresponding to the successive positions of the end-point, which satisfy the differential equation.

224. We shall show in the next Chapter that there must exist an enveloped space as described above, if the curve 01 is to offer a maximum or a minimum. There are exceptional cases which are to be treated separately.

We may at first assume the existence of such a space in order to make the essential points as clear as possible. We saw above that the function  $\bar{\mathfrak{E}}(x, y, \bar{p}, q, \bar{p}, \bar{q})$  along the whole curve for arbitrary values of  $\bar{p}, \bar{q}$  could not have different signs. From this we infer that in general  $\bar{\mathfrak{E}}(x, y, \bar{p}, q, \bar{p}, \bar{q})$  will not have values with different signs for other curves which satisfy the differential equation. The deviation in the directions of these curves from the position of the original curve, of course, lies within certain limits, and the corresponding  $\lambda$ 's vary sufficiently little from the  $\lambda$  of the original curve. This will certainly be true if the integral

$$\int_0^1 \left\{ F_1^0(x, y, \bar{p}_k, q_k) - \lambda F_1^0(x, y, \bar{p}_k, q_k) \right\} (1-k) dk$$

is everywhere different from zero along the first curve.

Excepting the case where the above integral becomes zero, we have as a further necessary condition that it must be possible to envelop the portion of curve 01 by a portion of surface, on the boundary of which the point 0 lies, so that within this portion of surface the function  $\bar{\mathfrak{E}}(x, y, \bar{p}, q, \bar{p}, \bar{q})$  does not have values with different signs along any of the curves that pass through 0, and lie within the portion of surface in question, it being assumed that they all satisfy the differential equation, and that the difference in value of  $\lambda$  is sufficiently small for all the curves. (See Art. 156).

225. The same considerations are also true for a point  $\bar{0}$  which lies before 0 along the same curve 01, so that then 01 lies wholly within the corresponding portion of surface, and our original enveloped space, including the point 1', may be so formed as to lie wholly within the space enveloped by this second surface.

Suppose that the integration of the above integrals begins now with the point  $\bar{0}$  instead of with the point 0 as before. Keeping our former notation, let the point  $0'$  correspond in space to 0 and join  $0'$  and  $1'$  by a regular curve which lies wholly within the enveloped space.

This curve is quite arbitrary and is subjected to the condition that if 2 is the projection of any point  $2'$  upon the  $xy$ -plane, the sum

$$I_{\bar{0}0}^{(1)} + \bar{I}_{02}^{(1)}$$

is equal to the length of the perpendicular projecting the point  $2'$ , where we use the notation  $I$  to represent an integral that is taken over a definite curve that satisfies the differential equation and  $\bar{I}$  one that is taken over an arbitrary curve, and where the indices represent the limits and the direction of the integration. A curve that satisfies the differential equation may be drawn from  $\bar{0}$  to every point  $2'$  of the curve in the enveloped space and this curve also with the exception of the point  $\bar{0}$  lies within the enveloped portion of space. These curves are to have the property, which after the assumptions is always possible, that beginning with  $\bar{0}1'$  the following curves are always variations of the preceding.

226. We regard the coördinates of the points of the projection of the arbitrary curve as functions of the length of arc counted from the point 1, and we consider the sum  $I_{\bar{0}2} + \bar{I}_{21}$ .

It follows from the fixed relation regarding the point in space that, wherever the point 2 may lie upon the curve  $021$ , we always have

$$I_{\bar{0}2}^{(1)} + \bar{I}_{21}^{(1)} = I_{\bar{0}1}^{(1)}.$$

There is consequently no variation in the integral  $I^{(1)}$ .

Let the length of the portion  $12$  increase by  $\sigma$ . The change thereby produced in  $I_{\bar{0}2} + \bar{I}_{21}$  is equal to

$$\mathfrak{E}(x_2, y_2, p_2, q_2, \bar{p}_2, \bar{q}_2) \sigma + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

where  $p_2, q_2$  are the direction-cosines of  $\bar{0}2$  at 2,  $\bar{p}_2, \bar{q}_2$  those of  $12$

at 2. Again let the length of arc 1 2 decrease by  $\sigma$ . The change thereby experienced in  $I_{02} + \bar{I}_{21}$  is equal to

$$-\mathfrak{E}(x_2, y_2, p_2, q_2, \bar{p}_2, \bar{q}_2) \sigma + \left( \sigma, \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2.$$

Since  $\xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}$  become indefinitely small with  $\sigma$ , the quantity  $\mathfrak{E}(x_2, y_2, p_2, q_2, \bar{p}_2, \bar{q}_2)$  represents the differential quotient of the sum  $I_{02} + \bar{I}_{21}$ , this sum being considered as a function of the length of arc 1 2 (see Art. 161).

If the point 2 coincides with 1, then is  $I_{02} + \bar{I}_{21} = I_{01}$ , and if 2 coincides with 0, we have

$$I_{02} + \bar{I}_{21} = I_{00} + \bar{I}_{01}.$$

Hence it follows :

1) If  $\mathfrak{E}$  along 021 is not positive and not everywhere zero, that

$$I_{01} > I_{00} + \bar{I}_{01};$$

2) If  $\mathfrak{E}$  along 021 is not negative and not everywhere zero, that

$$I_{01} < I_{00} + \bar{I}_{01}.$$

227. It will be shown in the next Chapter that we may assume the strips of surface enveloping  $\bar{0}01$  so narrow that the point  $\bar{0}$  may be joined with any other point within this enveloped space by one curve, and only one, which satisfies the differential equation. The curve must of course lie wholly within the enveloped space. This assumed, it follows that the integral  $I_{01}$  is identical with  $I_{001}$ , and further, that the integral  $I_{00}$  is identical with the portion of the integral  $I_{001}$ , which is taken over the portion of curve  $\bar{0}0$ .

We therefore have

$$\text{in case 1) } I_{01} > \bar{I}_{01};$$

$$\text{in case 2) } I_{01} < \bar{I}_{01}.$$

The maximal and minimal property of the curve that satisfies the differential equation is accordingly proved except in the case where along the whole curve the function  $\mathfrak{E}$  is zero.

228. We shall show that for the case where in the constructed realm the integral

$$\int_0^1 (F_1^{(0)} - \lambda F_1^{(1)}) (1-k) dk$$

is not zero, the function  $\mathfrak{E}$  cannot be zero along a whole curve between 0 and 1; or what is the same, that it is not possible everywhere along the curve to have

$$\bar{p}\bar{q} - \bar{p}q = 0.$$

After we have proved this theorem for a regular curve, we may extend it for a curve composed of regular portions; since, as has often been shown, sudden changes in the direction along the curve under consideration have no influence upon the deductions that have been drawn.

Finally, the same is also true for arbitrary curves which can be drawn between 0 and 1 and which lie sufficiently near the curve that satisfies the differential equation, but this is true in so far only as the two integrals have a meaning for these curves.

229. In a similar manner as was shown in Art. 165, the meaning of the integrals may be extended, if for these integrals are substituted sums of integrals which are taken over portions of regular curves that join a series of points on the curves under consideration. It remains then to show, when these sums approach finite fixed values by increasing the number of points and diminishing the distance between such points, that these limiting values are at the same time the values of the original integrals. Of course, the limiting value which is thus determined for the second integral  $I^{(1)}$  must be identical with the value that was prescribed for it.

If the integrals taken over an arbitrary curve have in this sense a definite value, it is clear that, for example in case of a maximum, the integral  $I^{(0)}$  taken over this curve cannot be greater than the integral taken over the curve which satisfies the differential equation. For we could form a curve out of regular portions of curve, the integral over which would be as little different from

the integral  $I^0$  as we wished, and consequently would also be greater than the integral taken over the curve which satisfies the differential equation. This is not possible after the hypothesis. Hence, that integral must be smaller than this one, as we may again show as follows: Let 2 be a point of the arbitrary curve sufficiently near 01, then we may draw two portions of curve which satisfy the differential equation, the one from 0 to 2 and the other from 2 to 1, the corresponding curves in space being  $0'2'$  and  $2'1'$ . Around  $0'2'$  and  $2'1'$  we may limit a portion of space in a similar manner as was done around  $0'1'$  and with the analogous properties. If the portion of space about  $0'1'$  is taken sufficiently small, the arbitrary curve will lie within the portion of space which envelops  $0'2'$  and  $2'1'$ .

The integral taken over this curve is, consequently, after what was given above, *not* greater than the integral taken over the two portions of curve which satisfy the differential equation; but this is smaller than  $I_{01}$ , and consequently also the integral of the arbitrary curve is smaller than  $I_{01}$ .

We must point out here a limitation which has been tacitly made: We traced the curve 021 in such a way that the corresponding curve in space lay in the portion of space defined above. Now, there may be curves 021 in a region about 01 taken arbitrarily small, such that the corresponding curves in space do not fall within the limited portion of space, and for such variations the maximal and minimal properties are not derived through our conclusions. Our proof has reference only to such variations in which the change of the value of the second integral becomes indefinitely small at corresponding points at the same time with the variation of the coördinates.

## CHAPTER XVIII.

## PROOF OF TWO THEOREMS WHICH HAVE BEEN ASSUMED IN THE PREVIOUS CHAPTER.

230. In the present Chapter proofs are given of the theorems:

1°. *That it is possible to construct a portion of space about a curve, which satisfies the differential equation of the problem, in such a way that it is always possible to join any point in this limited space and the initial point by one and only one curve which likewise satisfies the differential equation.*

2°. *The function  $\mathfrak{E}$  cannot vanish along an entire curve within such a portion of space.*

Let the coördinates  $x, y$  of a curve which satisfies the differential equation be expressed as functions of a quantity  $t$ . These functions contain three arbitrary constants: the two constants  $\alpha$  and  $\beta$  of integration and the constant  $\lambda$ . If then  $x, y$  and  $z$  are the coördinates of the corresponding point in space, we have

$$1) \quad x = \phi(t, \alpha, \beta, \lambda), \quad y = \psi(t, \alpha, \beta, \lambda), \quad z = \int_{t_0}^t F^{(1)}(x, y, x', y') dt,$$

where  $t=t_0$  corresponds to the point  $\bar{0}$ . By changing the three constants we have another curve in space. The requirement that the projection of this latter curve should go through the point  $\bar{0}$  gives two relations between the increments  $\alpha', \beta', \lambda', \tau_0$  of the constants  $\alpha, \beta, \lambda, t_0$ , where  $t_0 + \tau_0$  is the value of  $t$  in the new equation that corresponds to the point  $\bar{0}$ .

231. The equations of the new curve in space are :

$$2) \quad \begin{cases} x + \xi = \phi(t + \tau, a + a', \beta + \beta', \lambda + \lambda'), \\ y + \eta = \psi(t + \tau, a + a', \beta + \beta', \lambda + \lambda'), \\ z + \zeta = \int_{t_0 + \tau_0}^{t + \tau} F^{(1)} \left( x + \xi, y + \eta, x' + \frac{d\xi}{dt}, y' + \frac{d\eta}{dt} \right) dt; \end{cases}$$

and, if  $x_0, y_0$  are the coördinates of  $\bar{0}$ ,

$$3) \quad \begin{cases} x_0 = \phi(t_0 + \tau_0, a + a', \beta + \beta', \lambda + \lambda'), \\ y_0 = \psi(t_0 + \tau_0, a + a', \beta + \beta', \lambda + \lambda'). \end{cases}$$

These equations represent for sufficiently small values of  $\tau_0, a', \beta', \lambda'$ , which satisfy the last two equations, all curves in space which satisfy the differential equations and whose projection upon the  $x y$ -plane in its initial direction deviates very little from the initial direction of the projection of the original curve.

We may express  $\tau_0, a'$  and  $\beta'$  as power-series in  $\lambda'$  and the trigonometrical tangent of the angle which the two initial directions form with each other. If this tangent is denoted by  $k$ , we have, as in Art. 148,

$$4) \quad \begin{aligned} [\phi'(t_0)^2 + \psi'(t_0)^2] k &= [\psi'(t_0)\phi''(t_0) - \phi'(t_0)\psi''(t_0)]\tau_0 \\ &+ [\psi'(t_0)\phi_1'(t_0) - \phi'(t_0)\psi_1'(t_0)]a' \\ &+ [\psi'(t_0)\phi_2'(t_0) - \phi'(t_0)\psi_2'(t_0)]\beta' \\ &+ [\psi'(t_0)\phi_3'(t_0) - \phi'(t_0)\psi_3'(t_0)]\lambda' + [\tau_0, a', \beta', \lambda'], \end{aligned}$$

where  $\phi_3 = \frac{\partial \phi}{\partial \lambda}$ ,  $\psi_3 = \frac{\partial \psi}{\partial \lambda}$ .

232. Since the two curves are to go through the same initial point, we have further

$$5) \quad \begin{cases} 0 = \phi'(t_0)\tau_0 + \phi_1(t_0)\alpha' + \phi_2(t_0)\beta' + \phi_3(t_0)\lambda' + [\tau_0, \alpha', \beta', \lambda']_2, \\ 0 = \psi'(t_0)\tau_0 + \psi_1(t_0)\alpha' + \psi_2(t_0)\beta' + \psi_3(t_0)\lambda' + [\tau_0, \alpha', \beta', \lambda']_2. \end{cases}$$

The determinant of the linear terms on the right-hand side of the equations 4) and 5) is

$$\begin{vmatrix} \psi'(t_0)\phi''(t_0) - \phi'(t_0)\psi''(t_0) & \psi'(t_0)\phi_1'(t_0) - \phi'(t_0)\psi_1'(t_0) & \psi'(t_0)\phi_2'(t_0) - \phi'(t_0)\psi_2'(t_0) \\ \phi'(t_0) & \phi_1(t_0) & \phi_2(t_0) \\ \psi'(t_0) & \psi_1(t_0) & \psi_2(t_0) \end{vmatrix}$$

In this determinant write

$$6) \quad \theta_v(t) = \psi'(t)\phi_v(t) - \phi'(t)\psi_v(t). \quad (v=1, 2, 3)$$

If we multiply the second horizontal row by  $\psi''(t_0)$ , the third by  $-\phi''(t_0)$  and add both to the first, the determinant may then be written

$$\begin{vmatrix} 0 & , & \theta_1'(t_0) & , & \theta_2'(t_0) \\ \phi'(t_0) & , & \phi_1(t_0) & , & \phi_2(t_0) \\ \psi'(t_0) & , & \psi_1(t_0) & , & \psi_2(t_0) \end{vmatrix},$$

or,

$$\theta_2(t_0)\theta_1'(t_0) - \theta_1(t_0)\theta_2'(t_0).$$

This quantity is not zero, as we shall see later [see the third of equations 13) in Art. 237].

Hence we may express  $\tau_0, \alpha', \beta'$  as power-series in  $k, \lambda'$  so that for any pair of values  $k, \lambda'$ , which have been taken sufficiently small, there corresponds a curve in space.

From the differential equation it follows in a similar manner as was shown in Art. 149, that for one pair of values  $k, \lambda'$  there corresponds only one curve, and that every curve is completely determined through the initial point and the initial direction. We therefore, conclude, as in Art. 149, that the equations 4) and 5) afford us all the curves which are neighboring the original curve, which have the same end-point with it, and which satisfy the differential equation.

233. We have now to choose the constants in such a way that the new curve in space will go through a point  $x + \xi, y + \eta, z + \zeta$  which lies in the neighborhood of any point  $x, y, z$  situated on the old curve. If then we give to  $t$  a definite value and take sufficiently small values for  $\xi, \eta, \zeta$ , the following equations must be satisfied:

$$7) \left\{ \begin{array}{l} o = \phi'(t_0)\tau_0 + \phi_1(t_0)\alpha' + \phi_2(t_0)\beta' + \phi_3(t_0)\lambda' + (\tau_0, \alpha', \beta', \lambda')_2, \\ o = \psi'(t_0)\tau_0 + \psi_1(t_0)\alpha' + \psi_2(t_0)\beta' + \psi_3(t_0)\lambda' + (\tau_0, \alpha', \beta', \lambda')_2, \\ \xi = \phi'(t)\tau + \phi_1(t)\alpha' + \phi_2(t)\beta' + \phi_3(t)\lambda' + (\tau, \alpha', \beta', \lambda')_2, \\ \eta = \psi'(t)\tau + \psi_1(t)\alpha' + \psi_2(t)\beta' + \psi_3(t)\lambda' + (\tau, \alpha', \beta', \lambda')_2, \\ \zeta = \frac{\partial F^{(1)}}{\partial x'}\xi + \frac{\partial F^{(1)}}{\partial y'}\eta + \int_{t_0}^{t_1} G^{(1)}(y'\xi - x'\eta)dt + (\tau, \alpha', \beta', \lambda')_2. \end{array} \right.$$

But

$$y'\xi - x'\eta = \theta_1(t)\alpha' + \theta_2(t)\beta' + \theta_3(t)\lambda' + (\tau, \alpha', \beta', \lambda')_2.$$

Hence, if we write

$$8) \quad \Theta_v(t_0, t) = \int_{t_0}^{t_1} G^{(1)}\theta_v(t) dt, \quad (v=1, 2, 3)$$

it follows that

$$9) \quad \zeta = \frac{\partial F^{(1)}}{\partial x'}\xi + \frac{\partial F^{(1)}}{\partial y'}\eta + \Theta_1(t, t_0)\alpha' + \Theta_2(t, t_0)\beta' + \Theta_3(t, t_0)\lambda' + (\tau, \alpha', \beta', \lambda')_2.$$

If we substitute instead of  $\xi$  and  $\eta$  their power-series in  $\tau, \alpha', \beta', \lambda'$  in equation 9), the determinant of the linear terms on the right-hand side of equations 8) and 9) become after a slight transformation

$$10) \quad D(t_0, t) = \begin{vmatrix} \phi'(t_0), & o, & \phi_1(t_0), & \phi_2(t_0), & \phi_3(t_0) \\ \psi'(t_0), & o, & \psi_1(t_0), & \psi_2(t_0), & \psi_3(t_0) \\ o, & \phi'(t), & \phi_1(t), & \phi_2(t), & \phi_3(t) \\ o, & \psi'(t), & \psi_1(t), & \psi_2(t), & \psi_3(t) \\ o, & o, & \Theta_1(t_0, t), & \Theta_2(t_0, t), & \Theta_3(t_0, t) \end{vmatrix}.$$

We assume that this determinant does not vanish for arbitrary values of  $t$ . This case and the formulæ which would follow from it we leave as an exception for future investigation.

234. The first value of  $t$  after  $t_0$  for which  $D(t_0, t)$  vanishes we call the *conjugate* to  $t_0$ .

We see then that if the upper limit  $t_1$  of the integrals lies before the point that is conjugate to  $t_0$ , the curve can envelop a portion of space having the property desired.

Since in this case, if  $\xi, \eta, \zeta$  are chosen sufficiently small, one can always express  $\tau, \tau_0, \alpha', \beta', \lambda'$  as power-series in  $\xi, \eta, \zeta$ , and consequently can construct one and only one curve in space which satisfies the differential equation, which passes through the point  $\bar{0}$  and the point  $x + \xi, y + \eta, z + \zeta$  and which deviates in its position arbitrarily little from the original curve. To these different curves in space there correspond different functions  $D(t_0, t)$ . If, however, the curves lie sufficiently near the original curve, the functions  $D$  which correspond to them will not vanish for any point along them, so that through any point in a sufficiently small neighborhood of any point of these curves a curve starting from  $\bar{0}$  can be drawn which satisfies the differential equation.

235. It remains yet to be proved that, if the point conjugate to  $t_0$  lies between  $t_0$  and  $t_1$ , we cannot have a maximum or a minimum value of the integral.

Since the point  $\bar{0}$  can be chosen arbitrarily near 0 and since the point conjugate to  $t$  varies in a continuous manner with  $t_0$ , it is necessary only to show that  $t_1$  cannot lie between 0 and the point conjugate to it. We then will have proved everything except the case where  $t_1$  coincides with the point conjugate to 0. This case we must again leave for a special investigation, since the curve may or may not offer a maximum or a minimum. (cf. Art. 132.)

A rigorous proof of what has been said requires a close investigation of the function  $D(t_0, t)$ .

236. The curve in space which we had through variation of the constants is determined through the initial direction of its

projection at the point  $\bar{0}$  and through the differential equation  $G^{(0)} - (\lambda + \lambda') G^{(1)} = 0$ , which it must satisfy. From this the properties of the function  $D(t_0, t)$  may also be inferred.

We perform the changes which  $G^{(0)} - \lambda G^{(1)}$  suffers when  $\alpha, \beta, \lambda$  undergo the changes  $\alpha', \beta', \lambda'$ . The equation  $G^{(0)} - \lambda G^{(1)}$  must vanish for arbitrary values of  $\alpha', \beta', \lambda'$ .

We have

$$\Delta G = G^{(0)}(x + \xi, y + \eta) - G^{(0)}(x, y) - \lambda [G^{(1)}(x + \xi, y + \eta) - G^{(1)}(x, y)] - \lambda' G^{(1)}(x, y).$$

In a similar manner as was shown on page 133, formula (b), we have

$$G^{(0)}(x + \xi, y + \eta) - G^{(0)}(x, y) = F_2^{(0)}(y' \xi - x' \eta) - \frac{d}{dt} \left( F_1^{(0)} \frac{d(y' \xi - x' \eta)}{dt} \right) + \left( \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

$$G^{(1)}(x + \xi, y + \eta) - G^{(1)}(x, y) = F_2^{(1)}(y' \xi - x' \eta) - \frac{d}{dt} \left( F_1^{(1)} \frac{d(y' \xi - x' \eta)}{dt} \right) + \left( \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2,$$

and consequently

$$\Delta G = F_2(y' \xi - x' \eta) - \frac{d}{dt} \left( F_1 \frac{d(y' \xi - x' \eta)}{dt} \right) - \lambda' G^{(1)} + \left( \xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt} \right)_2.$$

The terms of the first dimension in the development of  $y' \xi - x' \eta$  in powers of  $\tau, \alpha', \beta', \lambda'$  are

$$\theta_1(t) \alpha' + \theta_2(t) \beta' + \theta_3(t) \lambda',$$

which we represent by  $\bar{w}$ . We then have

$$11) \quad \Delta G = - \frac{d}{dt} \left( F_1 \frac{d\bar{w}}{dt} \right) + F_2 \bar{w} - \lambda' G^{(1)}(x, y) + (\tau, \alpha', \beta', \lambda')_2.$$

Since this quantity must be zero for arbitrary values of  $\alpha', \beta', \lambda'$ , the coefficients of the individual terms in this expression when developed in power-series must be zero. If we limit ourselves to the linear terms, and use the functional sign for the function itself

when there can be no confusion, we have the following three differential equations:

$$12) \quad \begin{cases} F_1 \frac{d^2 \theta_1}{dt^2} + \frac{dF_1}{dt} \frac{d\theta_1}{dt} - F_2 \theta_1 = 0, \\ F_1 \frac{d^2 \theta_2}{dt^2} + \frac{dF_1}{dt} \frac{d\theta_2}{dt} - F_2 \theta_2 = 0, \\ F_1 \frac{d^2 \theta_3}{dt^2} + \frac{dF_1}{dt} \frac{d\theta_3}{dt} - F_2 \theta_3 - G^{(1)} = 0. \end{cases}$$

If we multiply the first of these equations by  $\theta_3$ , the second by  $-\theta_1$  and add the results, we have

$$\frac{d}{dt} \left[ F_1 \left( \theta_3 \frac{d\theta_1}{dt} - \theta_1 \frac{d\theta_3}{dt} \right) \right] = \theta_1 G^{(1)}.$$

Similarly, if we multiply the second equation by  $\theta_3$  and the third by  $-\theta_2$ , we have upon adding,

$$\frac{d}{dt} \left[ F_1 \left( \theta_3 \frac{d\theta_2}{dt} - \theta_2 \frac{d\theta_3}{dt} \right) \right] = \theta_2 G^{(1)}.$$

Finally, if we multiply the first equation by  $\theta_2$  and the second by  $-\theta_1$ , we have through addition,

$$\frac{d}{dt} \left[ F_1 \left( \theta_2 \frac{d\theta_1}{dt} - \theta_1 \frac{d\theta_2}{dt} \right) \right] = 0.$$

237. From these equations it follows that

$$13) \quad \begin{cases} \Theta_1(t_0, t) = \int_{t_0}^t \theta_1 G^{(1)} dt = \left[ F_1 \left( \theta_3 \frac{d\theta_1}{dt} - \theta_1 \frac{d\theta_3}{dt} \right) \right]_{t_0}^t, \\ \Theta_2(t_0, t) = \int_{t_0}^t \theta_2 G^{(1)} dt = \left[ F_1 \left( \theta_3 \frac{d\theta_2}{dt} - \theta_2 \frac{d\theta_3}{dt} \right) \right]_{t_0}^t, \\ F_1 \left( \theta_2 \frac{d\theta_1}{dt} - \theta_1 \frac{d\theta_2}{dt} \right) = C. \end{cases}$$

The constant  $C$  cannot be zero; for then we would have

$$\frac{d}{dt} \left( \log \theta_1 \right) = \frac{d}{dt} \left( \log \theta_2 \right),$$

$$\text{or } \theta_1 = C_1 \theta_2.$$

But, as is easily shown, the determinant  $D(t_0, t)$  may be brought to the form

$$14) \quad D(t_0, t) = \begin{vmatrix} \theta_1(t_0) & , & \theta_2(t_0) & , & \theta_3(t_0) \\ \theta_1(t) & , & \theta_2(t) & , & \theta_3(t) \\ \Theta_1(t_0, t) & , & \Theta_2(t_0, t) & , & \Theta_3(t_0, t) \end{vmatrix}.$$

If then  $\theta_1(t) = C_1 \theta_2(t)$ , it would also follow that  $\Theta_1(t_0, t) = C_1 \Theta_2(t_0, t)$ , and the determinant  $D(t_0, t)$  would vanish, since two vertical rows differ from each other only by a constant factor; and this is true for arbitrary values of  $t$ , which case we have excluded. Hence the constant  $C$  cannot be zero.

238. We next prove that the determinant  $D(t_0, t)$  changes sign, when it vanishes. We have

$$15) \quad \frac{d}{dt} D(t_0, t) = \begin{vmatrix} \theta_1(t_0) & , & \theta_2(t_0) & , & \theta_3(t_0) \\ \theta_1(t) & , & \theta_2(t) & , & \theta_3(t) \\ \frac{d}{dt} \Theta_1(t_0, t) & , & \frac{d}{dt} \Theta_2(t_0, t) & , & \frac{d}{dt} \Theta_3(t_0, t) \end{vmatrix} \\ + \begin{vmatrix} \theta_1(t_0) & , & \theta_2(t_0) & , & \theta_3(t_0) \\ \theta_1'(t) & , & \theta_2'(t) & , & \theta_3'(t) \\ \Theta_1(t_0, t) & , & \Theta_2(t_0, t) & , & \Theta_3(t_0, t) \end{vmatrix}.$$

Owing to 8), we have

$$\frac{d}{dt} \Theta_v(t_0, t) = G^{(v)} \theta_v(t).$$

Consequently the first of the determinants vanishes, leaving

$$\frac{d}{dt}D(t_0, t) = \begin{vmatrix} \Theta_1(t_0, t) & \Theta_2(t_0, t) & \Theta_3(t_0, t) \\ \theta_1(t_0) & \theta_2(t_0) & \theta_3(t_0) \\ \theta_1'(t) & \theta_2'(t) & \theta_3'(t) \end{vmatrix}.$$

We introduce the following notation:

$$16) \quad \begin{cases} f_1(t) = \theta_2(t_0)\theta_3(t) - \theta_3(t_0)\theta_2(t), \\ f_2(t) = \theta_3(t_0)\theta_1(t) - \theta_1(t_0)\theta_3(t), \\ f_3(t) = \theta_1(t_0)\theta_2(t) - \theta_2(t_0)\theta_1(t). \end{cases}$$

We can then write

$$D(t_0, t) = \sum_{v=1}^{v=3} f_v(t) \Theta_v(t_0, t),$$

$$\frac{d}{dt}D(t_0, t) = \sum_{v=1}^{v=3} f_v'(t) \Theta_v(t_0, t);$$

consequently

$$17) \quad f_3(t) \frac{d}{dt}D(t_0, t) - f_3'(t) D(t_0, t) = [f_3(t)f_1'(t) - f_1(t)f_3'(t)] \Theta_1(t_0, t) \\ + [f_3(t)f_2'(t) - f_2(t)f_3'(t)] \Theta_2(t_0, t),$$

or

$$18) \quad \frac{d}{dt} \left[ \frac{D(t_0, t)}{f_3(t)} \right] = \frac{[f_3(t)f_1'(t) - f_1(t)f_3'(t)] \Theta_1(t_0, t) + [f_3(t)f_2'(t) - f_2(t)f_3'(t)] \Theta_2(t_0, t)}{[f_3(t)]^2}.$$

239. The numerator of the right-hand side of the above expression is equal to  $F_1$  multiplied by the square of a certain expression, which we shall now determine.

Let us write

$$19) \quad E = \begin{vmatrix} \theta_1(t_0) & \theta_2(t_0) & \theta_3(t_0) \\ \theta_1(t) & \theta_2(t) & \theta_3(t) \\ \theta_1'(t) & \theta_2'(t) & \theta_3'(t) \end{vmatrix} = \theta_1'(t)f_1(t) + \theta_2'(t)f_2(t) + \theta_3'(t)f_3(t).$$

From 16) it follows at once that

$$19^a) \quad 0 = \theta_1(t_0)f_1(t) + \theta_2(t_0)f_2(t) + \theta_3(t_0)f_3(t),$$

and consequently

$$\begin{aligned} 20) \quad \theta_2(t_0)E &= [\theta_1'(t)\theta_2(t_0) - \theta_2'(t)\theta_1(t_0)]f_1(t) \\ &\quad + [\theta_3'(t)\theta_2(t_0) - \theta_2'(t)\theta_3(t_0)]f_3(t) \\ &= f_3(t)f_1'(t) - f_1(t)f_3'(t). \end{aligned}$$

Similarly, we have

$$21) \quad -\theta_1(t_0)E = f_3(t)f_2'(t) - f_2(t)f_3'(t).$$

Accordingly, the expression 18) may be written:

$$22) \quad \frac{d}{dt} \left[ \frac{D(t_0, t)}{f_3(t)} \right] = E \frac{[\theta_2(t_0)\Theta_1(t_0, t) - \theta_1(t_0)\Theta_2(t_0, t)]}{[f_3(t)]^2}.$$

But owing to the relations 13)

$$\begin{aligned} \Theta_1(t_0, t) &= \left[ F_1 \left\{ \theta_3(t)\theta_1'(t) - \theta_1(t)\theta_3'(t) \right\} \right]_{t_0}^t, \\ \Theta_2(t_0, t) &= \left[ F_1 \left\{ \theta_3(t)\theta_2'(t) - \theta_2(t)\theta_3'(t) \right\} \right]_{t_0}^t, \end{aligned}$$

it follows that

$$\begin{aligned} \theta_2(t_0)\Theta_1(t_0, t) - \theta_1(t_0)\Theta_2(t_0, t) &= \left[ F_1 \theta_3(t) \left\{ \theta_2(t_0)\theta_1'(t) - \theta_1(t_0)\theta_2'(t) \right\} \right]_{t_0}^t \\ &\quad - \left[ F_1 \theta_3'(t) \left\{ \theta_2(t_0)\theta_1(t) - \theta_1(t_0)\theta_2(t) \right\} \right]_{t_0}^t. \end{aligned}$$

240. Further we have

$$\begin{aligned} & \left[ F_1 \theta_3(t) \left\{ \theta_2(t_0) \theta_1'(t) - \theta_1(t_0) \theta_2'(t) \right\} \right]_{t_0}^t \\ &= \left[ F_1 \left\{ \theta_2(t) \theta_1'(t) - \theta_1(t) \theta_2'(t) \right\} \right]_{t_0}^t \theta_3(t_0) \\ &+ F_1 \theta_3(t) \left\{ \theta_2(t_0) \theta_1'(t) - \theta_1(t_0) \theta_2'(t) \right\} \\ &- F_1 \theta_3(t_0) \left\{ \theta_2(t) \theta_1'(t) - \theta_1(t) \theta_2'(t) \right\}. \end{aligned}$$

But owing to the third relation in 13) the expression  $F_1 \{ \theta_2(t) \theta_1'(t) - \theta_1(t) \theta_2'(t) \}$  is independent of  $t$ , so that the first term of the right-hand expression is zero, and consequently

$$23) \quad \theta_2(t_0) \Theta_1(t_0, t) - \theta_1(t_0) \Theta_2(t_0, t) = F_1(t) \begin{vmatrix} \theta_1(t_0), \theta_2(t_0), \theta_3(t_0) \\ \theta_1(t), \theta_2(t), \theta_3(t) \\ \theta_1'(t), \theta_2'(t), \theta_3'(t) \end{vmatrix} = F_1 E.$$

Hence the equation 22) becomes

$$24) \quad \frac{d}{dt} \left[ \frac{D(t_0, t)}{f_3(t)} \right] = \frac{F_1(t) E^2}{[f_3(t)]^2}.$$

241. Suppose that  $D(t_0, t)$  is zero of the  $k^{\text{th}}$  order for the value  $t=t'$  so that the development of  $D(t_0, t)$  begins with  $t-t'$  to the  $k^{\text{th}}$  power.

If then  $f_3(t)$  does not vanish for  $t=t'$ , the development of

$$f_3(t) \frac{d}{dt} D(t_0, t) - f_3'(t) D(t_0, t)$$

begins with the  $(k-1)^{\text{th}}$  power. But this expression is equal to  $F_1 E^2$ , and since according to our assumptions  $F_1$  does not become zero or infinity for any point within the interval  $t_0 \dots t_1$ , it is seen that  $F_1 E^2$  must begin with an even power. Hence  $k-1$  is an even integer, and consequently  $k$  is an odd integer, and therefore  $D(t_0, t)$  must change signs when it vanishes.

Suppose next that  $f_3(t)$  vanishes for  $t=t'$ , then  $f_3'(t)$  cannot vanish; for from the equations [see 16)]

$$\begin{aligned}\theta_1(t_0) \theta_2(t') - \theta_2(t_0) \theta_1(t') &= 0, \\ \theta_1(t_0) \theta_2'(t') - \theta_2(t_0) \theta_1'(t') &= 0,\end{aligned}$$

it would follow, if  $\theta_1(t_0)$  and  $\theta_2(t_0)$  are not simultaneously zero, that

$$\theta_1(t') \theta_2'(t') - \theta_2(t') \theta_1'(t') = 0.$$

But this equation, as also the simultaneous vanishing of  $\theta_1(t)$  and  $\theta_2(t)$  for the value  $t=t_0$ , contradicts the equation 13)

$$F_1[\theta_2(t) \theta_1'(t) - \theta_1(t) \theta_2'(t)] = C;$$

for, as we have seen,  $C$  is different from 0 and  $F_1$  is neither zero nor infinity.

Hence  $f_3(t)$  and  $f_3'(t)$  do not vanish simultaneously. If then  $f_3(t)$  vanishes for  $t=t'$ , the development of  $f_3(t)$  in powers of  $t-t'$  begins with the first power.

We may therefore write

$$\begin{aligned}D(t_0, t) &= A(t-t')^k + \dots, \\ f_3(t) &= c(t-t') + \dots\end{aligned}$$

It follows that the development of

$$f_3(t) \frac{d}{dt} D(t_0, t) - f_3'(t) D(t_0, t)$$

begins with the term

$$c A (k-1) (t-t')^k,$$

except when  $k=1$ , in which case the coefficient of this term is zero. In this case nothing has been shown, but see the next article.

For  $k=1$  it is evident that  $D(t_0, t)$  changes sign on vanishing.

242. We shall next show that if  $f_3(t)$  vanishes,  $E$  can be zero only when at the same time  $f_1(t)=0=f_2(t)$ . We saw in the preceding article that the quantities  $\theta_1(t_0)$  and  $\theta_2(t_0)$  cannot both be zero. If then  $\theta_1(t_0) \leq 0$ , it follows, from the relation [formula 21)]

$$f_3(t) f_2'(t) - f_2(t) f_3'(t) = -\theta_1(t_0) E$$

that, when  $E=0$  and  $f_3(t)=0$ , also  $f_2(t)=0$ , and also from

$$0 = \theta_1(t_0) f_1(t) + \theta_2(t_0) f_2(t) + \theta_3(t_0) f_3(t),$$

that  $f_1(t)=0$ .

Similarly, when  $\theta_2(t_0) \leq 0$ , it is seen from the equation

$$f_3(t) f_1'(t) - f_1(t) f_3'(t) = \theta_2(t_0) E,$$

that  $f_1(t)=0$ , if  $E=0=f_3(t)$ , and, consequently, also  $f_2(t)=0$ .

But if  $E$  does not vanish for  $t=t'$ , then  $k=0$ , and, consequently, also  $D(t_0, t)$  does not vanish for  $t=t'$ . We have thus shown that  $D(t_0, t)$  does not vanish for  $t=t'$ . It follows, therefore, that  $D(t_0, t)$  changes sign on vanishing except when we have simultaneously

$$f_1(t)=0=f_2(t)=f_3(t).$$

In this case it has not been proved whether it changes sign or does not. We must, consequently, consider each separate case for itself (see Art. 255).

243. If we assume that at least one of the quantities  $f_1(t)$ ,  $f_2(t)$ ,  $f_3(t)$  is different from zero, we can give the geometrical significance of conjugate points :

When the constants  $\alpha, \beta, \lambda$  are increased by  $\alpha', \beta', \lambda'$ , new curves in space are produced. The condition that one of these curves cuts the original curve in the point  $t''$  is [see equations 7) and 9)] expressed through the following equations :

$$25) \begin{cases} o = \phi'(t_0) \tau_0 + \phi_1(t_0) \alpha' + \phi_2(t_0) \beta' + \phi_3(t_0) \lambda' + (\tau_0, \alpha', \beta', \lambda')_2, \\ o = \psi'(t_0) \tau_0 + \psi_1(t_0) \alpha' + \psi_2(t_0) \beta' + \psi_3(t_0) \lambda' + (\tau_0, \alpha', \beta', \lambda')_2, \\ o = \phi'(t'') \tau + \phi_1(t'') \alpha' + \phi_2(t'') \beta' + \phi_3(t'') \lambda' + (\tau, \alpha', \beta', \lambda')_2, \\ o = \psi'(t'') \tau + \psi_1(t'') \alpha' + \psi_2(t'') \beta' + \psi_3(t'') \lambda' + (\tau, \alpha', \beta', \lambda')_2, \\ o = \Theta_1(t_0, t) \alpha' + \Theta_2(t_0, t) \beta' + \Theta_3(t_0, t) \lambda' + (\tau, \alpha', \beta', \lambda')_2, \end{cases}$$

or, if we eliminate  $\tau_0$  and  $\tau$ ,

$$\begin{aligned} o &= \theta_1(t_0) \alpha' + \theta_2(t_0) \beta' + \theta_3(t_0) \lambda' + (\alpha', \beta', \lambda')_2, \\ o &= \theta_1(t'') \alpha' + \theta_2(t'') \beta' + \theta_3(t'') \lambda' + (\alpha', \beta', \lambda')_2, \\ o &= \Theta_1(t_0, t) \alpha' + \Theta_2(t_0, t) \beta' + \Theta_3(t_0, t) \lambda' + (\alpha', \beta', \lambda')_2. \end{aligned}$$

The elimination of  $\alpha'$  and  $\beta'$  gives

$$26) \quad o = D(t_0, t'') \lambda' + (\lambda')_2,$$

or

$$o = D(t_0, t'') + (\lambda').$$

If  $D(t_0, t'')$  is different from zero, we may take for  $\lambda'$  a limit as small as we wish such that, for every  $\lambda'$  whose absolute value is less than the prescribed limit, we always have

$$|D(t_0, t'')| > |(\lambda')|.$$

Consequently no value of  $t''$  can be found which satisfies equation 26).

If then  $t'$  is a definite value of  $t''$  for which  $D(t_0, t'')$  is different from zero, there will be no value of  $t''$  within a certain interval  $t' - \tau_1, \dots, t' + \tau_2$  which satisfies the equation 26). Hence among all the curves in space for which  $\alpha', \beta', \lambda'$  have sufficiently small values there will be none which cuts the original curve in the neighborhood of  $t'$ .

It is quite different, however, if we take for  $t''$  an interval  $t' - \tau, \dots, t' + \tau$  which contains  $t'$ , the point conjugate to  $t_0$ , within which, therefore,  $D(t_0, t'') = 0$ . For then  $D(t_0, t'')$  has opposite

signs for  $t''=t'-\tau$  and  $t''=t'+\tau$ . Hence after an arbitrarily small value  $\tau$  has been fixed, we can always choose  $\lambda'$  so small that also

$$D(t_0, t'') + (\lambda')$$

has opposite signs for  $t''=t'-\tau$  and  $t''=t'+\tau$ , and consequently there will be within this interval a value of  $t''$  for which the equation

$$D(t_0, t'') + (\lambda') = 0$$

is satisfied.

Hence, if we limit an interval ever so small about the point conjugate to  $\bar{0}$  and take arbitrarily small upper limits for  $\alpha'$ ,  $\beta'$ ,  $\lambda'$ , then among the admissible curves there are always such which start from  $\bar{0}$  and cut the original curve within this interval. Indeed, if  $\alpha'$ ,  $\beta'$ ,  $\lambda'$  are less than a certain quantity, then all the curves in space, for which  $\alpha'$ ,  $\beta'$ ,  $\lambda'$  have values not greater than this fixed quantity and which go through the point  $\bar{0}$ , cut the original curve within this interval. This upper limit for  $\alpha'$ ,  $\beta'$ ,  $\lambda'$  becomes infinitely small at the same time with this interval, so that *the point conjugate to  $\bar{0}$  can be defined as the point which the points of intersection of neighboring curves approach.*

244. In a similar manner we may prove that a portion of space as small as we choose may be taken around a point of the curve in space which is not conjugate to  $\bar{0}$ , and that the points along the curves in space, which are conjugate to  $\bar{0}$ , do not lie within this limited portion of space, if  $\alpha'$ ,  $\beta'$ ,  $\lambda'$  are taken sufficiently small; but when we limit a portion of space as small as we wish about the point that is conjugate to  $\bar{0}$ , the points along the curves in space that are conjugate to  $\bar{0}$  will with sufficiently small  $\alpha'$ ,  $\beta'$ ,  $\lambda'$  all lie within this interval.

It also follows that, if in  $D(t_0, t)$  the quantity  $t_0$  varies in a continuous manner, the first value of  $t$ , for which  $D(t_0, t)$  vanishes, varies in a continuous manner. This follows at once from

$$D(t_0 + \tau, t) = D(t_0, t) + (\tau, t),$$

where  $(\tau, t)$  becomes infinitely small with  $\tau$  for every value of  $t$ .

For  $t=t'-\tau$  and  $t=t'+\tau$ , where  $t'$  is the point conjugate to the point  $t_0$ , the function  $D(t_0, t)$  has different signs, however

small  $\tau$  is; and if we take  $\tau$  sufficiently small, it follows that  $D(t_0, t) + (\tau, t)$  has different signs for  $t = t' - \tau$  and  $t = t' + \tau$  and must therefore vanish for some value of  $t$  within the interval  $t' - \tau \dots t' + \tau$ . The change in the conjugate point is consequently arbitrarily small for a sufficiently small increment in  $t_0$ .

245. We come next to the proof of the theorem that *a portion of curve which includes  $t_0$  and the point conjugate to it may always be so varied that  $\Delta I^0$  may be both positive and negative, while  $I^{(1)}$  remains unchanged.*

Let us write as in Arts. 180, 181:

$$\bar{\xi} = \epsilon \xi + \epsilon_1 \xi_1,$$

$$\bar{\eta} = \epsilon \eta + \epsilon_1 \eta_1.$$

We have accordingly

$$\Delta I^{(1)} = \epsilon \int_{t_0}^{t_1} G^{(1)} w dt + \epsilon_1 \int_{t_0}^{t_1} G^{(1)} w_1 dt + (\epsilon, \epsilon_1)_2.$$

Now choose  $w$  so that

$$27) \quad \int_{t_0}^{t_1} G^{(1)} w dt = 0.$$

Then from the condition that  $\Delta I^{(1)} = 0$ , it follows that we may express  $\epsilon_1$  as a power-series in  $\epsilon$  which begins with a power higher than the first in  $\epsilon$ .

Hence (see Art. 180), it follows that

$$\bar{w} = \epsilon w + (\epsilon)_2,$$

and, from Art. 189, that

$$\Delta I^0 = \frac{\epsilon^2}{2} \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + F_2 w^2 \right\} dt + (\epsilon)_3;$$

or, what is the same thing :

$$28) \quad \Delta I^{(0)} = \frac{k \epsilon^2}{2} \int_{t_0}^{t_1} w^2 dt + \frac{\epsilon^2}{2} \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + (F_2 - k) w^2 \right\} dt + (\epsilon)_3,$$

where  $k$  is an arbitrarily small quantity over which we have yet a choice.

246. We shall now show that if  $t_1$  lies beyond the point conjugate to  $t_0$ , the absolute value of  $k$  may be chosen so small that besides satisfying the condition

$$\int_{t_0}^{t_1} G^{(1)} w dt = 0,$$

the quantity  $w$  will satisfy also the condition

$$29) \quad \int_{t_0}^{t_1} \left\{ F_1 \left( \frac{dw}{dt} \right)^2 + (F_2 - k) w^2 \right\} dt = 0$$

without being everywhere zero. If  $\epsilon$  is chosen sufficiently small, which we are always able to do, it is then seen that the quantity

$\Delta I^{(0)}$  has the same sign as  $k \epsilon^2 \int_{t_0}^{t_1} w^2 dt$ , and consequently the same sign as  $k$ , which may be either positive or negative.

Since  $w$  vanishes for  $t_0$  and  $t_1$ , and since

$$\frac{d}{dt} \left[ F_1 w \frac{dw}{dt} \right] = F_1 \left( \frac{dw}{dt} \right)^2 + w \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right),$$

it follows that instead of 29), we may write:

$$\int_{t_0}^{t_1} \left\{ - \frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) + (F_2 - k) w \right\} w dt = 0;$$

and instead of this equation and the equation

$$\int_{t_0}^{t_1} G^1 w dt = 0,$$

we may write the two equations:

$$30) \quad \left\{ \begin{array}{l} \int_{t_0}^{t_1} \left\{ -\frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) + (F_2 - k) w - e_3 G^1 \right\} w dt = 0, \\ \int_{t_0}^{t_1} G^{(1)} w dt = 0, \end{array} \right.$$

where  $e_3$  is a quantity independent of  $t$ .

247. Now let  $\theta_1(t, k)$ ,  $\theta_2(t, k)$ ,  $\theta_3(t, k)$  be three functions of  $t$  which satisfy the three differential equations:

$$31) \quad \left\{ \begin{array}{l} \frac{d}{dt} \left( F_1 \frac{d}{dt} \theta_1(t, k) \right) - (F_1 - k) \theta_1(t, k) = 0, \\ \frac{d}{dt} \left( F_1 \frac{d}{dt} \theta_2(t, k) \right) - (F_2 - k) \theta_2(t, k) = 0, \\ \frac{d}{dt} \left( F_1 \frac{d}{dt} \theta_3(t, k) \right) - (F_3 - k) \theta_3(t, k) - G^1 = 0. \end{array} \right.$$

It follows from the theory of differential equations that for a series of values of  $t$  for which  $F_1$  is neither zero nor infinite  $\theta_1(t, k)$ ,  $\theta_2(t, k)$ ,  $\theta_3(t, k)$  differ from the three functions  $\theta_1(t)$ ,  $\theta_2(t)$ ,  $\theta_3(t)$  by quantities which become infinitely small at the same time with  $k$ .

Again, let  $t'$  be the point conjugate to  $t_0$ , and write for the stretch from  $t_0$  to  $t'$ , where  $t''$  is a point situated before the point  $t'$ ,

$$w = e_1 \theta_1(t, k) + e_2 \theta_2(t, k) + e_3 \theta_3(t, k),$$

and for the stretch from  $t'$  to  $t_1$  let

$$w = 0.$$

It is clear that  $w$  is not everywhere zero unless  $e_1=o=e_2=e_3$ , since owing to the differential equations 31) which  $\theta_1(t, k)$ ,  $\theta_2(t, k)$ ,  $\theta_3(t, k)$  satisfy, a linear relation for the series of values of  $t$  can exist only if for these values  $G^{(1)}=o$ , a case which we excluded (Art. 180).

248. The quantity  $w$  satisfies the differential equation

$$\frac{d}{dt} \left( F_1 \frac{dw}{dt} \right) - (F_2 - k) w - G^{(1)} = o.$$

It must also satisfy the additional conditions that  $w=o$  for  $t_0$  and for  $t''$ , and that

$$\int_{t_0}^{t_1} G^{(1)} w dt = o.$$

But we have

$$\int_{t_0}^{t''} G^{(1)} w dt = e_1 \int_{t_0}^{t''} G^{(1)} \theta_1(t, k) dt + e_2 \int_{t_0}^{t''} G^{(1)} \theta_2(t, k) dt + e_3 \int_{t_0}^{t''} G^{(1)} \theta_3(t, k) dt.$$

If we write

$$\int_{t_0}^{t''} G^{(1)} \theta_v(t, k) dt = \Theta_v(t_0, t'', k), \quad (v=1, 2, 3)$$

then from what was seen above, the functions  $\Theta_v(t_0, t'', k)$  differ from  $\Theta_v(t_0, t'')$  by a quantity which becomes infinitely small with  $k$ .

The conditions which remain to be fulfilled are:

$$32) \quad \begin{cases} o = e_1 \theta_1(t_0, k) + e_2 \theta_2(t_0, k) + e_3 \theta_3(t_0, k), \\ o = e_1 \theta_1(t'', k) + e_2 \theta_2(t'', k) + e_3 \theta_3(t'', k), \\ o = e_1 \Theta_1(t_0, t'', k) + e_2 \Theta_2(t_0, t'', k) + e_3 \Theta_3(t_0, t'', k). \end{cases}$$

The determinant of these equations differs from  $D(t_0, t'')$  by a quantity which becomes infinitely small with  $k$  (Art. 237).

For  $t'' = t' - k$  and  $t'' = t' + k$  the quantity  $D(t_0, t'')$  has different signs, and consequently we may take  $k$  so small that the determinant of the equation 32) has different signs for  $t'' = t' - k$  and  $t'' = t' + k$  and consequently vanishes for a value of  $t''$  situated between  $t' - k$  and  $t' + k$ .

We may therefore take  $t''$  along the curve before  $t_1$  in such a way that the equations 32) are satisfied by values of  $e_1, e_2$  and  $e_3$  which are not all zero.

If then, returning to equation 28),  $\epsilon$  is chosen sufficiently small, it follows that  $\Delta I^0$  has the sign of  $k$  and since this is arbitrary, there are among the admissible variations of the curves those for which  $I^0$  has a negative increment and also those for which the increment of  $I^0$  is positive.

The portion of curve 01 cannot therefore extend beyond the point which is conjugate to 0. If we exclude the case where 1 coincides exactly with the point that is conjugate to 0, it follows that 1 must lie before the point that is conjugate to 0. We may then choose  $\bar{0}$  so near to 0 that 1 lies also before the point that is conjugate to  $\bar{0}$ . Along such a portion of curve the function  $D(t_0, t)$  does not vanish and consequently we may envelop such a portion of curve in a portion of space which has the required properties.

249. It only remains, excluding exceptional cases, to show that the function  $\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q})$  cannot vanish along an entire curve within the portion of space defined above.

If we exclude the possibility of the integral

$$\int_0^1 \{F_1^0(x, y, p_k, q_k) - \lambda F_1^1(x, y, p_k, q_k)\} (1-k) dk$$

becoming zero along a portion of the curve in question, then for  $\mathfrak{E}$  to vanish, it is necessary that  $\bar{p}q - p\bar{q} = 0$  along the whole curve; that is, the direction of the projection of the arbitrary curve in space must coincide at every point with the direction of the projection of the curve that satisfies the differential equation.

If  $x, y, z, x_1, y_1, z_1$  are the coördinates of the two curves expressed as functions of their lengths of arc  $s=t$  and  $s'=t'$ , say, then at the point in question we must have

$$x=x_1, y=y_1, \frac{dx}{dt} = \frac{dx_1}{dt'}, \frac{dy}{dt} = \frac{dy_1}{dt'}.$$

But since

$$z = \int_{t_0}^t F^{(1)}(x, y, x', y') dt,$$

and

$$z_1 = \int_{t'_0}^{t'} F^{(1)}(x_1, y_1, x'_1, y'_1) dt',$$

it follows also that

$$\frac{dz}{dt} = \frac{dz_1}{dt'},$$

i. e., the two curves in space have at every point also the same direction.

The quantities

$$\left\{ \begin{array}{l} \phi(t_1 + \tau, \alpha + \alpha', \beta + \beta', \lambda + \lambda'), \\ \psi(t_1 + \tau, \alpha + \alpha', \beta + \beta', \lambda + \lambda'), \\ \int_{t_0}^{t_1 + \tau} F^{(1)}(\phi, \psi, \phi', \psi') dt \end{array} \right.$$

represent the coördinates of the neighboring curve in space in the neighborhood of the point  $t_1$ .

The point  $t_1$  is now taken as any arbitrary point. If in the above expressions we consider  $\tau, \alpha', \beta', \lambda'$  as functions of a quantity  $k$  which become infinitely small with  $k$ , then for successive values of  $k$  these expressions are the coördinates of the points of a cer-

tain curve which goes through  $t_1$ ; and indeed every curve that passes through  $t_1$  can be expressed in this manner, if the functions of  $k$  are suitably chosen.

250. If now there is to be a curve in space along which  $\mathfrak{E}=0$ , then its direction at the point  $\tau, a', \beta', \lambda'$ , as we saw above, must coincide at this point with the direction of the curve which satisfies the differential equation determined through  $a', \beta', \lambda'$ .

The direction-cosines of the latter curve are proportional to the following quantities:

$$a) \begin{cases} \phi'(t_1 + \tau, a + a', \beta + \beta', \lambda + \lambda'), \\ \psi'(t_1 + \tau, a + a', \beta + \beta', \lambda + \lambda'), \\ \left( \frac{\partial F^{(1)}}{\partial x'} \phi'(t_1 + \tau, a + a', \beta + \beta', \lambda + \lambda') + \frac{\partial F^{(1)}}{\partial y'} \psi'(t_1 + \tau, a + a', \beta + \beta', \lambda + \lambda') \right); \end{cases}$$

and those of the first curve to:

$$\left[ \phi'(t_1 + \tau, a + a', \beta + \beta', \lambda + \lambda') \frac{d\tau}{dk} + \frac{d\phi}{dk}; \text{ where in } \frac{d\phi}{dk} \text{ only } a', \beta' \text{ and } \lambda' \right. \\ \left. \text{are to be considered as dependent upon } k. \right.$$

$$\left[ \psi'(t_1 + \tau, a + a', \beta + \beta', \lambda + \lambda') \frac{d\tau}{dk} + \frac{d\psi}{dk}, \right. \\ \left( \frac{\partial F^{(1)}}{\partial x'} \phi' + \frac{\partial F^{(1)}}{\partial y'} \psi' \right) \frac{d\tau}{dk} \\ \left. + \int_{t_0}^{t_1 + \tau} \left\{ \frac{\partial F^{(1)}}{\partial x} \frac{d\phi}{dk} + \frac{\partial F^{(1)}}{\partial y} \frac{d\psi}{dk} + \frac{\partial F^{(1)}}{\partial x'} \frac{d\phi'}{dk} + \frac{\partial F^{(1)}}{\partial y'} \frac{d\psi'}{dk} \right\} dt, \right.$$

where in  $\frac{d\psi}{dk}, \frac{d\phi'}{dk}, \frac{d\psi'}{dk}$  only  $a', \beta', \lambda'$  are to be considered as dependent upon  $k$ , the increment due to  $\tau$  being already explicitly expressed. If we integrate by parts the expression that stands under the integral sign of the last of the above quantities, and take into consideration the definitions 8) of Art. 233, it is seen that the

direction-cosines of the arbitrary curve are proportional to the quantities

$$b) \left\{ \begin{array}{l} \phi' \frac{d\tau}{dk} + \frac{d\phi}{dk}, \\ \psi' \frac{d\tau}{dk} + \frac{d\psi}{dk}, \\ \left( \frac{\partial F^{(1)}}{\partial x'} \phi' + \frac{\partial F^{(1)}}{\partial y'} \psi' \right) \frac{d\tau}{dk} + \Theta_1(t_0, t_1 + \tau) \frac{d\alpha'}{dk} + \Theta_2(t_0, t_1 + \tau) \frac{d\beta'}{dk} \\ \quad + \Theta_3(t_0, t_1 + \tau) \frac{d\lambda'}{dk} + \frac{\partial F^{(1)}}{\partial x'} \frac{d\phi}{dk} + \frac{\partial F^{(1)}}{\partial y'} \frac{d\psi}{dk}. \end{array} \right.$$

251. If the direction of the two curves are to coincide at the point in question, then the three minors formed from the quantities *a*) and *b*) must vanish. But these minors are identical with the minors formed from the quantities

$$\phi' \quad , \quad \psi' \quad , \quad \frac{\partial F^{(1)}}{\partial x'} \phi' + \frac{\partial F^{(1)}}{\partial y'} \psi',$$

$$\frac{d\phi}{dk} \quad , \quad \frac{d\psi}{dk} \quad , \quad \frac{\partial F^{(1)}}{\partial x'} \frac{d\phi}{dk} + \frac{\partial F^{(1)}}{\partial y'} \frac{d\psi}{dk} + \Theta_1 \frac{d\alpha'}{dk} + \Theta_2 \frac{d\beta'}{dk} + \Theta_3 \frac{d\lambda'}{dk}.$$

Accordingly, the three quantities of the first row are proportional to the corresponding quantities of the second row.

If we make  $k=0$ , the above quantities become

$$\phi'(t, \alpha, \beta, \lambda), \psi'(t, \alpha, \beta, \lambda), \frac{\partial F^{(1)}}{\partial x'} \phi'(t, \alpha, \beta, \lambda) + \frac{\partial F^{(1)}}{\partial y'} \psi'(t, \alpha, \beta, \lambda);$$

$$\phi_1(t, \alpha, \beta, \lambda) \left( \frac{d\alpha'}{dk} \right)_0 + \phi_2(t, \alpha, \beta, \lambda) \left( \frac{d\beta'}{dk} \right)_0 + \phi_3(t, \alpha, \beta, \lambda) \left( \frac{d\lambda'}{dk} \right)_0,$$

$$\psi_1 \left( \frac{d\alpha'}{dk} \right)_0 + \psi_2 \left( \frac{d\beta'}{dk} \right)_0 + \psi_3 \left( \frac{d\lambda'}{dk} \right)_0,$$

$$\begin{aligned} & \frac{\partial F^{(1)}}{\partial x'} \left[ \phi_1 \left( \frac{d\alpha'}{dk} \right)_0 + \phi_2 \left( \frac{d\beta'}{dk} \right)_0 + \phi_3 \left( \frac{d\lambda'}{dk} \right)_0 \right] + \frac{\partial F^{(1)}}{\partial y'} \left[ \psi_1 \left( \frac{d\alpha'}{dk} \right)_0 + \psi_2 \left( \frac{d\beta'}{dk} \right)_0 + \psi_3 \left( \frac{d\lambda'}{dk} \right)_0 \right] \\ & + \Theta_1(t_0, t) \left( \frac{d\alpha'}{dk} \right)_0 + \Theta_2(t_0, t) \left( \frac{d\beta'}{dk} \right)_0 + \Theta_3(t_0, t) \left( \frac{d\lambda'}{dk} \right)_0. \end{aligned}$$

Hence, if we let  $\rho$  denote the factor of proportionality, we have

$$33) \quad \begin{cases} o = \phi_1 \left( \frac{d\alpha'}{dk} \right)_0 + \phi_2 \left( \frac{d\beta'}{dk} \right)_0 + \phi_3 \left( \frac{d\lambda'}{dk} \right)_0 + \rho \phi', \\ o = \psi_1 \left( \frac{d\alpha'}{dk} \right)_0 + \psi_2 \left( \frac{d\beta'}{dk} \right)_0 + \psi_3 \left( \frac{d\lambda'}{dk} \right)_0 + \rho \psi', \\ o = \Theta_1(t_0, t) \left( \frac{d\alpha'}{dk} \right)_0 + \Theta_2(t_0, t) \left( \frac{d\beta'}{dk} \right)_0 + \Theta_3(t_0, t) \left( \frac{d\lambda'}{dk} \right)_0, \end{cases}$$

where the third equation is reduced to this form by the application of the other two.

252. Since the curve which satisfies the differential equation must pass through the point  $t_0$ , we must in virtue of equations 5) and 6) have the relation

$$o = \theta_1(t_0) \alpha' + \theta_2(t_0) \beta' + \theta_3(t_0) \lambda',$$

and from this it follows that

$$34) \quad o = \theta_1(t_0) \left( \frac{d\alpha'}{dk} \right)_0 + \theta_2(t_0) \left( \frac{d\beta'}{dk} \right)_0 + \theta_3(t_0) \left( \frac{d\lambda'}{dk} \right)_0.$$

Eliminate  $\rho$  from the first two equations in 33) and write for the differences that appear their values in terms of the  $\theta$ 's defined by the relations 6).

The determinant of the resulting equation, of the last of the equations 33), and of equation 34) is identical with  $D(t_0, t_1)$ . [See formula 14), Art. 237]. Hence if  $D(t_0, t_1)$  does not vanish, these equations have no other solution except

$$\left( \frac{d\alpha'}{dk} \right)_0 = \left( \frac{d\beta'}{dk} \right)_0 = \left( \frac{d\lambda'}{dk} \right)_0 = o.$$

The same conclusions may also be drawn from any small value of  $k$  to which the values  $\tau, \alpha', \beta', \lambda'$  correspond; there enters here instead of the quantity  $D(t_0, t_1)$  the quantity

$$D(t_0, t_1 + \tau, \alpha + \alpha', \beta + \beta', \lambda + \lambda').$$

Since this determinant for sufficiently small values of  $\tau, \alpha', \beta', \lambda'$  is different from zero, it also follows that

$$\frac{d\alpha'}{dk} = \frac{d\beta'}{dk} = \frac{d\lambda'}{dk} = 0.$$

But the quantities  $\alpha', \beta', \lambda'$  do not vary with  $k$  and are consequently zero, since they are zero for  $k=0$ ; this means that the curve along which the function  $\mathfrak{E}$  is to be zero must coincide with the original curve which satisfies the differential equation. It is then no variation of this curve and the arbitrariness of the quantities  $\bar{p}, \bar{q}$  which is essential to the meaning of the function  $\mathfrak{E}$  is entirely lost.

If  $D(t_0, t_1)=0$ , then  $t_1$  would be the point conjugate to  $t_0$ ; and since this would be true for every point  $t_1$  of the arbitrary curve, it would follow that the arbitrary curve was formed from the points that are the conjugates of the initial point and would consequently lie without or at least on the boundary of the portion of space under consideration.

It is seen that there is no curve within our assumed portion of space along which the function  $\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q})$  vanishes everywhere. The purport of this Chapter is thus completed. It has also been shown that the conditions necessary and sufficient for the existence of a maximum or a minimum are in the Theory of Relative Maxima and Minima the analogues of those enumerated in Art. 174.

253. We shall now finish the proof of the maximal and minimal properties of the two problems already considered, viz., the *isoperimetrical problem* and the *problem of finding the curve whose center of gravity lies lowest*.

In the case of the isoperimetrical problem we have

$$F = -yx' - \lambda \sqrt{x'^2 + y'^2},$$

$$\frac{\partial F}{\partial x'} = -y - \frac{\lambda x'}{\sqrt{x'^2 + y'^2}} = -y - \lambda p,$$

$$\frac{\partial F}{\partial y'} = -\lambda \frac{y'}{\sqrt{x'^2 + y'^2}} = -\lambda q,$$

$$F_1 = \frac{-\lambda}{(\sqrt{x'^2 + y'^2})^3},$$

$$x = a + \lambda \cos t, \quad y = b + \lambda \sin t,$$

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q}) = \lambda(\bar{p}p + q\bar{q} - 1).$$

In these expressions  $\lambda$  is a positive constant; further,  $x'$  and  $y'$  do not vanish simultaneously at any point of the curve, so there is no exceptional case which requires a special investigation. The expression  $\bar{p}p + q\bar{q}$  is the cosine of the angle between the two directions  $p, q$  and  $\bar{p}, \bar{q}$ , so that  $\mathfrak{E}$  has for no point and no direction a positive value. As we have already seen, this is one of the requirements for a maximum.

254. Let two points 0 and 1 be connected by the arc of a circle of given length, which together with a fixed curve, that joins the two points, incloses a surface-area. The integral taken over the whole periphery is represented by (see Art. 191)

$$\int_{t_0}^{t_1} -yx' dt,$$

and it is required to prove that one cannot connect the two points by a curve of the same length which includes a greater area with the fixed curve. The proof is immediate as soon as the following is shown. If an arbitrary curve of the prescribed length is drawn between the two points, then we may draw through any point 2 of this curve the arc of a circle which also goes through 0 and which has the same length as the portion of the arbitrary curve situated between 0 and 2.

If we let the point 2 traverse the arbitrary curve, the successive arcs of circles are variations of one another and their lengths differ indefinitely little from one another. If the point 2 is sufficiently near the initial point, the corresponding arc of circle becomes the arc of circle for which the maximal property is to be proved.

This is all done as soon as we stipulate that each arc of circle is to be traversed once only and is to be constructed as indicated in Art. 229. Then indeed there is only one arc of circle having the given length that can be laid between the points 0 and 2. The arcs of circles corresponding to the successive lengths are variations of one another, and their lengths at corresponding points differ indefinitely little from one another, and consequently the arcs of circles, if the point 2 coincides with  $t_1$ , pass in a *continuous manner* into the original arc of circle drawn between 0 and 1.

255. Regarding the determinant  $D(t_0, t)$ , we have here

$$x = a + \lambda \cos t, \quad y = b + \lambda \sin t,$$

$$\theta_1(t) = \lambda \cos t, \quad \theta_2(t) = \lambda \sin t, \quad \theta_3(t) = \lambda, \quad G^{(1)} = \frac{1}{\lambda},$$

$$\Theta_1(t_0, t) = \sin t - \sin t_0, \quad \Theta_2(t_0, t) = \cos t_0 - \cos t, \quad \Theta_3(t_0, t) = t - t_0,$$

$$f_1(t) = \lambda^2(\sin t_0 - \sin t), \quad f_2(t) = \lambda^2(\cos t - \cos t_0), \quad f_3(t) = \lambda^2 \sin(t - t_0).$$

From these expressions we have

$$\begin{aligned} D(t_0, t) &= \begin{vmatrix} \lambda \cos t_0 & , & \lambda \sin t_0 & , & \lambda \\ \lambda \cos t & , & \lambda \sin t & , & \lambda \\ \sin t - \sin t_0 & , & \cos t_0 - \cos t & , & t - t_0 \end{vmatrix} \\ &= \lambda^2 \left\{ (t - t_0) \sin(t - t_0) + 2 \cos(t - t_0) - 2 \right\} \\ &= 4 \lambda^2 \sin \frac{t - t_0}{2} \left\{ \frac{t - t_0}{2} \cos \frac{t - t_0}{2} - \sin \frac{t - t_0}{2} \right\}. \end{aligned}$$

It is seen from this that the first time after  $t_0$  that  $D(t_0, t)$  vanishes, is for the value  $t = t_0 + 2\pi$ ; consequently  $t = t_0 + 2\pi$  is the point conjugate to the initial-point.

In reality, if we consider the initial and the end-point coinciding so that the curve satisfying the differential equation is a complete circle, then this curve does no longer offer a maximum, at least, in the sense that with every arbitrarily small variation of the curve the variation would be smaller; since we could slide at pleasure the curve congruent to itself, and therefore vary the curve without altering the perimeter or the surface-area.

It is interesting to observe that a case appears in this problem which could not be decided in the general treatment; namely, where  $f_1(t)$ ,  $f_2(t)$ ,  $f_3(t)$  simultaneously vanish with  $D(t_0, t)$  (see Art. 242).

In reality for  $t=t_0+2\pi$ , we have

$$D(t_0, t)=0, f_1(t)=f_2(t)=f_3(t)=0.$$

Nevertheless,  $D(t_0, t)$  changes sign when it passes through zero; for the vanishing of  $D(t_0, t)$  is effected by making the factor  $\sin \frac{t-t_0}{2}$  zero. But this factor changes sign, while the second factor retains its sign for  $t=t_0+2\pi$ .

256. In the problem of *finding the curve whose center of gravity lies lowest*, we had (Art. 216)

$$F=(y-\lambda) \sqrt{x'^2+y'^2},$$

$$\frac{\partial F}{\partial x'}=(y-\lambda) \frac{x'}{\sqrt{x'^2+y'^2}}=(y-\lambda)p; \frac{\partial F}{\partial y'}=(y-\lambda) \frac{y'}{\sqrt{x'^2+y'^2}}=(y-\lambda)q;$$

$$F_1=\frac{y-\lambda}{(\sqrt{x'^2+y'^2})^3};$$

$$x=a+\beta t, \quad y=\lambda+\frac{\beta}{2}(e^t+e^{-t});$$

$$\mathfrak{E}(x, y, p, q, \bar{p}, \bar{q})=(y-\lambda) [1-(p\bar{p}+q\bar{q})].$$

We saw that  $y-\lambda>0$ , and further  $x'$  and  $y'$  do not vanish simultaneously at any point. Consequently  $F_1$  is everywhere different from 0 and  $\infty$ . Since  $p\bar{p}+q\bar{q}$  represents the cosine of the angle between the two directions  $p, q$  and  $\bar{p}, \bar{q}$ , its absolute value cannot exceed unity, and in general is less than unity, so that the function  $\mathfrak{E}$  is nowhere negative, as must be the case for a minimum. We have already seen in Art. 219, if the length of arc is sufficiently great, that between two arbitrarily given points one curve and only one may be drawn which satisfies the differential equation. It then follows that there can be no conjugate points and consequently the catenary in its whole trace has the desired minimal property.

257. That there are no conjugate points is also seen from the consideration of the determinant  $D(t_0, t)$ . For we have

$$x = a + \beta t, \quad y = \lambda + \frac{\beta}{2}(e^t + e^{-t}),$$

$$\theta_1(t) = \frac{\beta}{2}(e^t - e^{-t}), \quad \theta_2(t) = \frac{\beta t}{2}(e^t - e^{-t}) - \frac{\beta}{2}(e^t + e^{-t}),$$

$$\theta_3(t) = -\beta, \quad G^{(1)} = \frac{4}{\beta(e^t + e^{-t})^2},$$

$$\Theta_1(t_0, t) = 2 \frac{e^t + e^{-t} - (e^{t_0} + e^{-t_0})}{(e^t + e^{-t})(e^{t_0} + e^{-t_0})}, \quad \Theta_2(t_0, t) = 2 \frac{t_0(e^t + e^{-t}) - t(e^{t_0} - e^{-t_0})}{(e^t + e^{-t})(e^{t_0} + e^{-t_0})},$$

$$\Theta_3(t_0, t) = -2 \frac{e^{-t_0}(e^t + e^{-t}) - e^{-t}(e^{t_0} + e^{-t_0})}{(e^t + e^{-t})(e^{t_0} + e^{-t_0})}.$$

From these quantities we have

$$D(t_0, t) = \frac{-\beta^2}{(e^t + e^{-t})(e^{t_0} + e^{-t_0})} \text{ multiplied by the determinant}$$

$$\begin{vmatrix} e^{t_0} - e^{-t_0} & , & t_0(e^{t_0} - e^{-t_0}) - (e^{t_0} + e^{-t_0}), & 2 \\ e^t - e^{-t} & , & t(e^t - e^{-t}) - (e^t + e^{-t}), & 2 \\ e^t + e^{-t} - (e^{t_0} + e^{-t_0}), & t_0(e^t - e^{-t}) - t(e^{t_0} + e^{-t_0}), & e^{-t_0}(e^t + e^{-t}) - e^{-t}(e^{t_0} + e^{-t_0}) \end{vmatrix};$$

or

$$\begin{aligned} D(t_0, t) &= -2\beta^2 \left\{ e^{t-t_0} + e^{-(t-t_0)} - \frac{t-t_0}{2}(e^{t-t_0} - e^{-(t-t_0)}) - 2 \right\} \\ &= 2\beta^2 \left( e^{\frac{t-t_0}{2}} - e^{-\frac{t-t_0}{2}} \right) \left\{ \frac{t-t_0}{2} \left( e^{\frac{t-t_0}{2}} + e^{-\frac{t-t_0}{2}} \right) - \left( e^{\frac{t-t_0}{2}} - e^{-\frac{t-t_0}{2}} \right) \right\}. \end{aligned}$$

The equation  $D(t_0, t) = 0$ , or

$$\frac{t-t_0}{2} \left( e^{\frac{t-t_0}{2}} + e^{-\frac{t-t_0}{2}} \right) - \left( e^{\frac{t-t_0}{2}} - e^{-\frac{t-t_0}{2}} \right) = 0$$

has no real root except  $t = t_0$ ; that is, there exists no point conjugate to the point  $t = t_0$ .

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